Prudential Monetary Policy

Ricardo J. Caballero and Alp Simsek*

December 18, 2019

Abstract

Should monetary policy have a *prudential* dimension? That is, should policymakers raise interest rates to rein in financial excesses during a boom? We theoretically investigate this question using an aggregate demand model with asset price booms and financial speculation. In our model, monetary policy affects financial stability through its impact on *asset prices*. Our main result shows that, when macroprudential policy is imperfect, small doses of *prudential monetary policy (PMP)* can provide financial stability benefits that are equivalent to tightening leverage limits. PMP reduces asset prices during the boom, which softens the asset price crash when the economy transitions into a recession. This mitigates the recession because higher asset prices support leveraged, high-valuation investors' balance sheets. An alternative intuition is that PMP raises the interest rate to create room for monetary policy to react to negative asset price shocks. The policy is most effective when there is extensive speculation and leverage limits are neither too tight nor too slack. With shadow banks, whether PMP "gets in all the cracks" or not depends on the constraints faced by shadow banks. When shadow banks face no leverage limits, PMP can still replicate the benefits of macroprudential policy, but PMP is less effective (like macroprudential policy) because shadow banks respond by increasing their leverage.

JEL Codes: E00, E12, E21, E22, E30, E40, G00, G01, G11
Keywords: Speculation, leverage, aggregate demand, business cycle, effective lower bound, monetary policy, regulation, macroprudential policies, leaning against the wind, shadow banks.

Dynamic link to the most recent draft:
https://www.dropbox.com/s/nmzrbx964e12yus/PMP_public.pdf?dl=0

---

1. Introduction

Should monetary policy have a *prudential* dimension? That is, should policymakers raise interest rates, or delay a cut, to rein in financial excesses during a boom? This question has occupied the minds of central bankers and monetary policy researchers for decades. At present, there are two dominant views. The *fully-separable* view contends that monetary policy should focus exclusively on its traditional mandate while delegating financial stability concerns to macroprudential policy (see, e.g., Weidmann (2018); Svensson (2018)). The *non-separable* view argues that, in practice, macroprudential policy might be insufficient to deal with financial excesses since its tools are limited and inflexible (see, e.g., Stein (2014); Gourio et al. (2018)). This debate has spawned a growing literature investigating the costs and benefits of *prudential monetary policy (PMP)*. In this paper, we provide a new rationale for PMP, and we show that under appropriate circumstances it can be as effective as macroprudential policy. This equivalence is useful since, as highlighted by the non-separable view, monetary policy in practice is significantly nimbler than macroprudential policy when responding to cyclical fluctuations.[1]

PMP has obvious costs: it slows down the economy and leads to inefficient factor utilization during the boom. The benefits are less well understood. One of the main arguments for PMP is the *asset price channel:* monetary policy can mitigate the asset price boom and therefore make the subsequent crash smaller and less costly (see, e.g., Borio (2014); Adrian and Liang (2018)). This view is supported by evidence that monetary policy has a sizable, nearly immediate impact on asset prices. Despite its potential importance, there is little formal analysis on how the asset price channel of PMP works and whether (or when) it improves social welfare. We fill this gap by developing an aggregate demand model with asset price booms and speculation.

In our model, the economy transitions from a boom with high asset prices into a recession with low asset prices. The boom features *financial speculation*—investors with heterogeneous valuations trading risky financial assets amongst themselves. We focus on speculation among investors with heterogeneous beliefs (optimists and pessimists), but similar insights apply if speculation is driven by other forces such as heterogeneous risk tolerances (e.g., banks and households). The recession features *interest rate frictions*—factors that might constrain how the risk-free rate adjusts after a shock. We focus on the zero lower bound, but our mechanism applies for other constraints that prevent policymakers from cutting interest rates during recessions.

These ingredients make optimists' wealth share a key state variable for the economy. In particular, when optimists have more wealth in the recession, they push up asset prices and aggregate demand, softening the recession. However, individual optimists who take on leverage during the boom (and pessimists who lend to them) do not internalize the welfare effects of

---

[1]Adrian et al. (2017) summarize the results from a tabletop exercise conducted by the Federal Reserve that "aimed at confronting Federal Reserve Bank presidents with a plausible, albeit hypothetical, macro-financial scenario that would lend itself to macroprudential considerations...From among the various tools considered, tabletop participants found many of the prudential tools less attractive due to implementation lags and limited scope of application...Monetary policy came more quickly to the fore as a financial stability tool than might have been thought before the exercise."
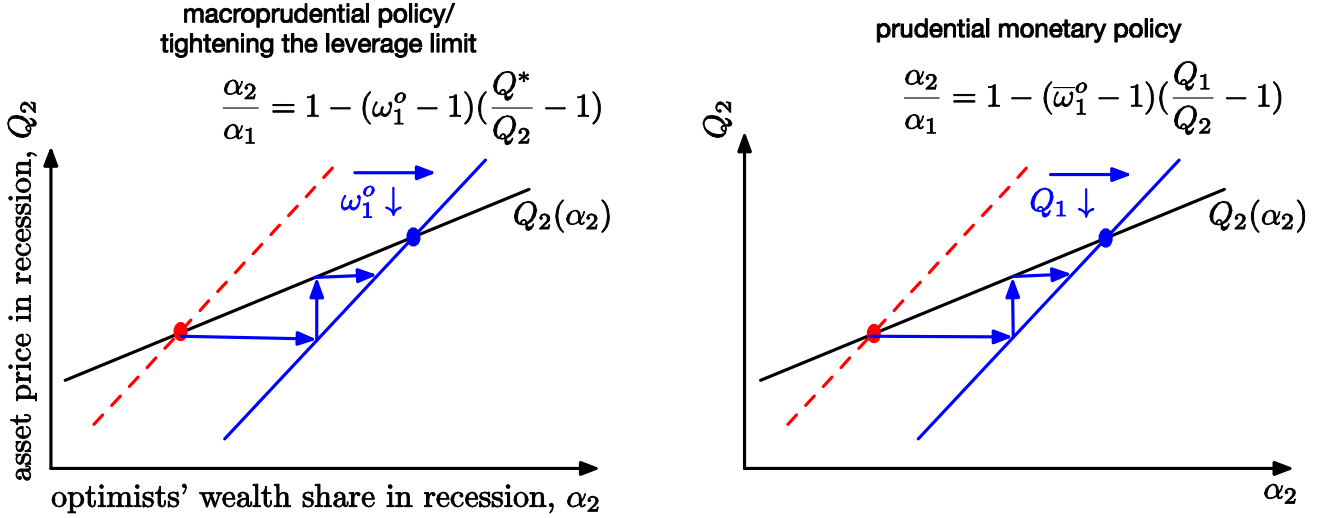
macroprudential policy/ tightening the leverage limit

$$\frac{\alpha_2}{\alpha_1} = 1 - (\omega_1^o - 1)(\frac{Q^*}{Q_2} - 1)$$

$\omega_1^o \downarrow$

$Q_2(\alpha_2)$

asset price in recession, Q_2

optimists' wealth share in recession, α_2

prudential monetary policy

$$\frac{\alpha_2}{\alpha_1} = 1 - (\overline{\omega}_1^o - 1)(\frac{Q_1}{Q_2} - 1)$$

$Q_1 \downarrow$

$Q_2(\alpha_2)$

Figure 1: Graphical illustration of the relations that determine optimists' wealth share and the asset price in recession, (α_2, Q_2). The left (resp. right) panel illustrates the effect of macroprudential policy (resp. PMP).

optimists' wealth losses during the recession, which motivates policy interventions. Macroprudential policy is in theory the ideal tool for disciplining optimists' risk taking, but in practice is imperfect. Our main result shows when PMP can effectively reduce optimists' risk exposure.

To illustrate this result, we introduce some notation and relations (we provide microfoundations in the main text). Specifically, let $s = 1$ and $s = 2$ denote the *boom* and the *recession* states, respectively. The economy is set in continuous time and transitions from the boom state to the recession state according to a Poisson process. Let α_s and Q_s denote optimists' wealth share and the price of capital (asset price) in state s, respectively. In the recession state $s = 2$, the price of capital is an increasing function of optimists' wealth share:

$$Q_2 = Q_2(\alpha_2). \tag{1}$$

In the boom state $s = 1$, optimists choose an above-average leverage ratio, $\omega_1^o > 1$. Therefore, if there is a transition to the recession state, their wealth share declines. Specifically, we have,

$$\frac{\alpha_2}{\alpha_1} = 1 - (\omega_1^o - 1)\left(\frac{Q_1}{Q_2} - 1\right), \tag{2}$$

where $Q_1/Q_2 > 1$ captures the magnitude of the price decline after the transition. Note that this equation also describes an increasing relation between optimists' wealth share, α_2, and the price of capital in the recession, Q_2 (since $\omega_1^o > 1$). Given a boom wealth share α_1, the equilibrium pair (α_2, Q_2) corresponds to the intersection of two increasing relations (1) and (2), similar to Kiyotaki and Moore (1997). Figure 1 provides a graphical representation of these relations.

In this framework, aggregate demand is an increasing function of asset prices, so monetary

policy can be described in terms of its effect on asset prices. As a benchmark, suppose the monetary authority sets interest rates in the boom to achieve asset prices and aggregate demand consistent with potential output, $Q_1 = Q^*$. In the recession, monetary policy is constrained, so asset prices and aggregate demand fall short of potential output, $Q_2 < Q^*$. A larger wealth share for optimists, α_2, increases asset prices and aggregate demand and softens the recession. This effect is an *aggregate demand externality*, which provides a rationale for prudential policies that improve optimists' wealth share in the recession, α_2.

Eq. (2) suggests that there are two prudential channels policymakers can use to increase α_2. First consider macroprudential policy that reduces optimists' leverage ratio, ω_1^o. This policy increases α_2 by reducing optimists' exposure to a given asset price decline, Q^*/Q_2. Second, suppose instead that optimists' leverage ratio is fixed, $\omega_1^o = \overline{\omega}_1^o$, due to either financial frictions, self-imposed limits, or binding macroprudential policy, and consider PMP that reduces asset prices during the boom, $Q_1 < Q^*$. This policy increases α_2 by decreasing the size of the asset price decline, Q_1/Q_2, for a given level of optimists' exposure. Figure 1 shows that these two policies can achieve the same allocations, illustrating the logic behind our main result.

Moreover, as we shall see in the formal derivation, PMP lowers asset prices, $Q_1 < Q^*$, by setting the interest rate higher than the benchmark with conventional output stabilization ("rstar"). Thus, an equivalent intuition for our main result is that PMP raises the interest rate to create room for monetary policy to react to negative asset price shocks. This interpretation would not apply in the standard New Keynesian model where the severity of the recession depends only on the *level* of interest rates. In our model, the *path* of interest rates also matters because optimists' balance sheet is a key state variable that is affected by *changes* in asset prices.

PMP has two drawbacks relative to macroprudential policy. First, optimists' leverage ratio has to be at least somewhat constrained, $\omega_1^o = \overline{\omega}_1^o$. While this is likely to be the case in practice —due to financial frictions, self-imposed limits, or binding macroprudential policy— our model provides a cautionary note for environments in which optimists' constraints are loose. In particular, in the extreme case in which optimists are fully unconstrained, their leverage ratio adjusts to completely undo the prudential effects of monetary policy. That is, once ω_1^o adjusts, α_2 does not depend on Q_1. The intuition is that, since optimists perceive smaller risks after transition to a recession, they increase their leverage ratio. This result illustrates that PMP is more effective when optimists face tighter leverage constraints.

Second, even when monetary policy achieves the same prudential objectives as macroprudential policy, it is more costly because it lowers asset prices during the boom, $Q_1 < Q^*$, which reduces factor utilization below the efficient level. However, in a neighborhood of the price level that ensures efficient factor utilization (Q^*), these negative welfare effects are second order. On the other hand, the beneficial effects of softening the recession are first order. Our main result formalizes this insight and establishes that (when optimists are subject to some leverage limit) the first-order welfare effects of PMP are *exactly the same* as the effects of tightening the leverage limit directly. Put differently, *for small policy changes, PMP is as effective as macroprudential*

policy. PMP increases unemployment in a booming economy, which has negligible costs, and reduces unemployment during a recession, which has sizeable benefits.

This discussion illustrates how our main result may apply beyond our particular model of recessions. For example, suppose the recession features no interest rate frictions, but there are *financial frictions* and *fire-sale prices* that increase in *experts'* wealth share. Suppose experts take on leverage during the boom to increase the size of their investments (as in Lorenzoni (2008)). The analogues of Eqs. (1) and (2) apply in this setting. Hence, as long as experts' leverage is constrained, PMP would improve experts' balance sheets in the recession and increase welfare. In this alternative setup, the policy would increase welfare by mitigating *fire-sale externalities*, whereas in our model PMP internalizes aggregate demand externalities.

We also characterize the optimal monetary policy in our environment and establish three comparative statics results. First, the planner utilizes PMP more when the leverage limit (or macroprudential policy) is at an intermediate level. Intuitively, when the limit is too loose, PMP is not worthwhile because it requires a large decline in Q_1 to push optimists against their constraints. Naturally, when the limit is already too tight, further tightening via PMP is not beneficial. These two extreme cases illustrate that macroprudential policy and PMP can be *complements* as well as *substitutes*. Second, as expected, the planner utilizes PMP more when she perceives a greater probability of transitioning into a recession. Finally, the planner utilizes PMP more when investors have greater disagreements about the risk of a recession. This result highlights that the policy is not driven by high asset prices *per se* (which is addressed by conventional monetary policy objectives) but by the financial speculation associated with episodes that concentrate risks on optimists' (or banks') balance sheets.

Finally, one of the main practical concerns with prudential policies is the presence of "shadow banks"—lightly regulated high-valuation agents who can circumvent regulatory constraints. Stein (2013) noted that in these environments PMP might have an advantage over macroprudential policy "because it gets in all of the cracks." We extend our analysis to consider shadow banks—optimists who are not subject to regulatory leverage limits. We find that whether PMP is more effective than macroprudential policy depends on the nature of the leverage limits faced by shadow banks. Even if shadow banks circumvent the regulatory leverage limit, they might still be constrained due to financial frictions or self-imposed limits. In this case, shadow banks and regular banks both face binding leverage limits, so our earlier analysis applies and implies that PMP is indeed more effective than macroprudential policy. We also analyze the other extreme case in which shadow banks are fully unconstrained. In this case, PMP can still replicate the financial stability benefits of macroprudential policy; however, both policies are weaker than when there are no shadow banks. The policies are weaker because of general equilibrium feedbacks: shadow banks (when unconstrained) respond to the stabilizing benefits of either policy by increasing their leverage and risk taking.

Literature review. Our paper is part of a large literature that investigates the effect of

monetary policy on financial stability. Adrian and Liang (2018) provide a recent survey (see also Smets (2014)). As they note, easy monetary policy can generate financial vulnerabilities by fueling credit growth, exacerbating the maturity mismatch of financial intermediaries, and inflating asset prices. Our paper focuses on the asset-price channel, which is underexplored.

One strand of the literature emphasizes that loose monetary policy can reduce risk premia during the boom by exacerbating the "reach for yield" (see, e.g., Rajan (2006); Maddaloni and Peydró (2011); Borio and Zhu (2012); Morris and Shin (2014); Lian et al. (2018); Acharya and Naqvi (2018)). In our model, monetary policy does not directly affect the risk premium—it affects asset prices mainly through the traditional discount rate channel. Nonetheless, we find a role for PMP because the reduction in asset prices during the boom softens the asset price crash after transition to recession. Our channel is stronger (and it operates through the same key equations) if, as suggested by empirical evidence, monetary policy also affects the risk premium during the boom (e.g., Bernanke and Kuttner (2005); Hanson and Stein (2015); Gertler and Karadi (2015); Gilchrist et al. (2015)).

Our paper complements the literature emphasizing the credit channel. A number of papers show that monetary policy can affect financial stability by influencing credit growth or leverage. Woodford (2012) articulates this channel using a New Keynesian framework (that builds upon Curdia and Woodford (2010)) in which loose monetary policy increases the leverage of financial institutions (or borrowers), which in turn increases the probability of a crisis (by assumption). We show that monetary policy can also affect financial stability by influencing asset prices during the boom. Moreover, our model does not require a financial crisis: there are benefits if the economy transitions into a plain-vanilla recession (in which monetary policy is constrained). Hence, our theoretical findings suggest that quantitative analyses that rely purely on the credit channel and financial crises likely underestimate the benefits of PMP.[2]

In our model, PMP causes an output gap during the boom, which generates a second-order welfare loss (for small changes in policy), and mitigates the output gap during the recession, which generates a first-order welfare gain. Kocherlakota (2014) and Stein (2014) derive similar insights by assuming that the Fed uses a quadratic loss function to penalize deviations of unemployment from its target. They show that targeting financial stability fits naturally into the Fed's dual mandate. Our model provides a microfoundation for their key assumption that accommodative monetary policy exacerbates financial vulnerability.

Our paper is part of a growing theoretical literature that analyzes the interactions between macroprudential and monetary policies in environments with aggregate demand externalities (see, e.g., Korinek and Simsek (2016); Farhi and Werning (2016); Rognlie et al. (2018)).[3] Most

---

[2]A growing empirical literature has documented that rapid credit growth is associated with more frequent and more severe financial crises (e.g., Borio and Drehmann (2009); Jordà et al. (2013)). Recent work uses the empirical estimates from this literature to calibrate Woodford-style models and quantify the costs and benefits of PMP. Svensson (2017); IMF (2015) argue that the costs of this policy exceed the benefits, whereas Gourio et al. (2018); Adrian and Liang (2018) find mixed effects.

[3]Several papers analyze the interaction of macroprudential and monetary policies but focus on other frictions (e.g., Stein (2012); Collard et al. (2017); Martinez-Miera and Repullo (2019)). A vast literature theoretically

of these papers conclude that financial stability issues are best addressed with macroprudential policy. We depart from this literature by assuming that macroprudential policy is constrained, and we find a role for monetary policy that interacts with macroprudential policy. We also investigate the asset price channel, whereas Korinek and Simsek (2016) and Farhi and Werning (2016) focus on credit.Rognlie et al. (2018) analyze investment and show that incorporating this ingredient would strengthen our main result. When alternative policies are imperfect, PMP can be used to reduce investment during the boom. PMP creates pent-up investment demand that raises investment, asset prices, and aggregate demand during the recession.

Finally, although our mechanism is more general, our specific model with belief disagreements and speculation is related to a large finance literature (e.g., Lintner (1969); Miller (1977); Harrison and Kreps (1978); Scheinkman and Xiong (2003); Fostel and Geanakoplos (2008); Geanakoplos (2010); Simsek (2013a,b); Iachan et al. (2015); Cao (2017); Heimer and Simsek (2018)). Similar to Caballero and Simsek (2017), we analyze speculation when aggregate demand can influence output due to interest rate rigidities. We depart from our earlier work by assuming that financial markets are incomplete due to exogenous leverage limits (see Remark 4). This assumption ensures that monetary policy affects the extent of speculation.

In Section 2 we introduce the basic environment and provide a partial characterization of the equilibrium. In Section 3, we characterize the equilibrium in the recession state and illustrate the aggregate demand externalities that motivate policy interventions. In Section 4, we characterize the equilibrium in the boom state for a benchmark case without PMP and illustrate how macroprudential policy can improve welfare. In Section 5, we introduce PMP and establish our main results regarding its (local) equivalence with macroprudential policy. In Section 6, we characterize the optimal PMP in our setting and establish its comparative statics. In Section 7, we add "shadow banks" to our framework and analysis. Section 8 concludes and is followed by several appendices that contain omitted derivations and proofs.

2. Environment and equilibrium

In this section we introduce our general dynamic environment. We then provide a definition and a partial characterization of the equilibrium. In subsequent sections we further analyze this equilibrium under different assumptions about monetary policy.

Potential output and risk premium shocks. The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production, capital. Let $k_{t,s}$ denote the capital stock at time t in the aggregate state $s \in S$.

The rate of capital utilization is endogenous and denoted by $\eta_{t,s} \in [0, 1]$. When utilized at

---

investigates macroprudential policy but doesn't focus on nominal rigidities or monetary policy (see, e.g., Dávila and Korinek (2017) and the references therein).

this rate, $k_{t,s}$ units of capital produce

$$A\eta_{t,s}k_{t,s} \tag{3}$$

units of the consumption good. The capital stock follows the process

$$\frac{dk_{t,s}/dt}{k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right). \tag{4}$$

The depreciation function $\delta\left(\eta_{t,s}\right)$ is increasing. Hence, Eqs. (3) and (4) illustrate that utilizing capital at a higher rate allows the economy to produce more current output at the cost of faster depreciation and slower output growth. Without nominal rigidities, there is an optimal level of capital utilization denoted by η^*, which we characterize in the subsequent analysis. With nominal rigidities, the economy may operate below this level of utilization, $\eta_{t,s} \leq \eta^*$, due to aggregate demand shortages.

Eq. (4) also illustrates that the expected growth rate of capital (before depreciation) is given by g_s, which is an exogenous parameter. The states, $s \in S$, differ only in terms of g_s. For simplicity, we assume there are three states, $s \in \{1, 2, 3\}$. The economy starts in state $s = 1$. While in states $s \in \{1, 2\}$, the economy transitions into state $s' \equiv s + 1$ according to a Poisson process that we describe below. Once the economy reaches $s = 3$, it stays there forever.

We assume the parameters satisfy $g_2 < \min(g_1, g_3)$. We envision a scenario in which the economy starts in the boom state with a relatively high growth rate, eventually enters a recession state with a low growth rate, then returns to an absorbing recovery state with a high growth rate. Accordingly, we refer to states $1, 2$, and 3 as "the boom," "the recession," and "the recovery," respectively. For analytical tractability, we focus on a single business cycle. Figure 2 illustrates the timeline of events for a particular realization of state transitions.

Remark 1 (Broadening the interpretation of expected growth fluctuations)**.** *We view the changes in the expected growth rate, g_s, as a device to capture more broadly "time-varying risk premia": that is, fluctuations in risky asset prices that are unrelated to short-run fundamentals (i.e., the current supply-determined output level). In Caballero and Simsek (2017), we formalize this intuition by showing that (in a two period model) changes in g_s generate the same effect on asset prices and economic activity as changes in risk or risk aversion. A large literature documents that time-varying risk premia are a pervasive phenomenon in financial markets (see Cochrane (2011); Campbell (2014) for recent reviews).*

Transition probabilities and belief disagreements. We let $\lambda_s^i > 0$ denote investor i's belief about the Poisson transition probability from state s into state $s' = s + 1$. Since state $s = 3$ is an absorbing state, we have $\lambda_3^i = 0$ for each i. For the remaining states, we assume there are two types of investors, $i \in \{o, p\}$. Type o investors are "optimists," and type p investors are "pessimists." We denote the difference between perceived transition probabilities for optimists
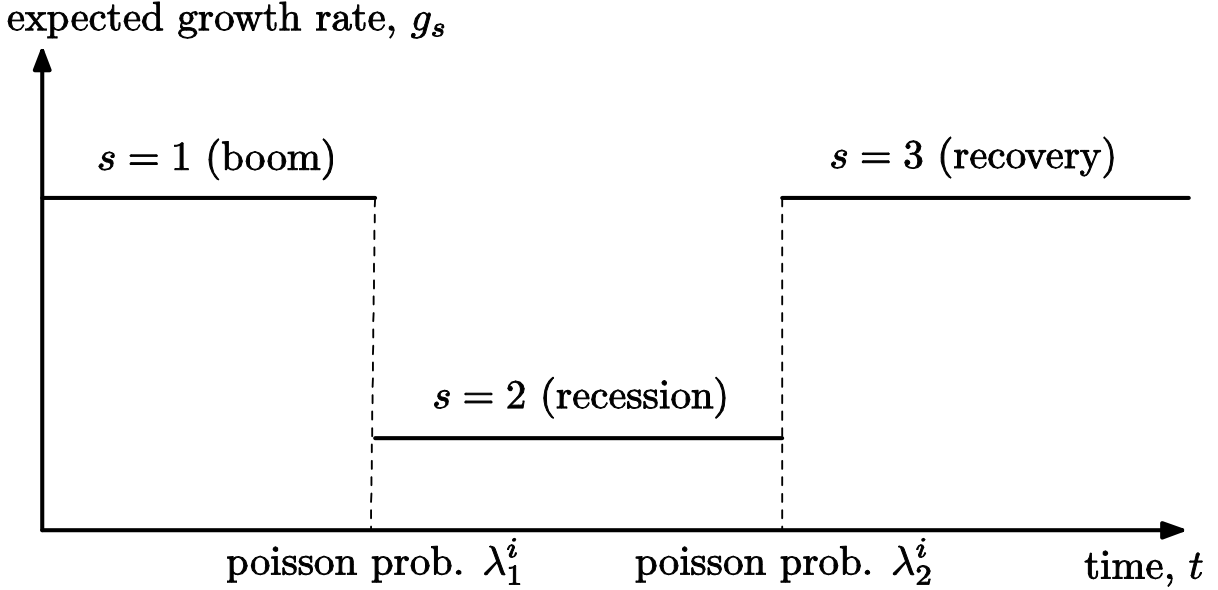
expected growth rate, g_s

Figure 2: The timeline of events.

and pessimists by

$$\Delta \lambda_s = \lambda_s^o - \lambda_s^p.$$

We assume the belief differences satisfy:

Assumption 1. $\Delta \lambda_1 < 0$ and $\Delta \lambda_2 > 0$.

When the economy is in the boom state $s = 1$, optimists assign a smaller transition probability to the recession state $s = 2$. When the economy is in the recession state, they assign a greater transition probability to the recovery state $s = 3$.

Remark 2 (Broadening the interpretation of disagreements). *We view disagreements about transition probabilities as a convenient modeling device to capture heterogeneous asset valuations. The key aspects of "optimists" is that they value risky assets more than "pessimists," so that: (i) during the boom, they take on leverage, and (ii) during the recession, they increase risky asset prices. These aspects would be the same with other modeling devices such as heterogeneous risk aversion or Knightian uncertainty. Consequently, we can also think of "optimists" as banks (or institutional investors) that are more risk tolerant and less Knightian than households or pension funds ("pessimists").*

Menu of financial assets. There are two types of financial assets. First, there is a market portfolio that represents a claim on all output (which accrues to production firms as earnings). We let $Q_{t,s} k_{t,s}$ denote the price of the market portfolio, so $Q_{t,s}$ is the price per unit of capital. We let $r_{t,s}$ denote the instantaneous expected return on the market portfolio conditional on no transition. Second, there is a risk-free asset in zero net supply. We denote its instantaneous return by $r_{t,s}^f$.

In Caballero and Simsek (2017), we allow for Arrow-Debreu securities that enable investors to trade the transition risk. In this paper, we assume financial markets are incomplete and thus investors speculate by adjusting their position on the market portfolio, i.e., changing their leverage ratio (see also Remark 4).

Market portfolio price and return. Absent state transitions, the price of capital $Q_{t,s}$ follows an endogenous, deterministic path. Using Eq. (4), the growth rate of the price of the market portfolio is given by

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}},$$

where we use the notation $\dot{X} \equiv dX/dt$. Consequently, the return of the market portfolio absent state transitions can be written as

$$r_{t,s} = \frac{y_{t,s}}{Q_{t,s}k_{t,s}} + g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}}. \tag{5}$$

Here, $y_{t,s}$ denotes the endogenous level of output at time t. Therefore, the first term captures the "dividend yield" component of return. The second term captures the capital gain conditional on no transition, which reflects the expected growth of capital and its price.

Portfolio choice. Investors are identical except for their beliefs about state transitions, λ_s^i. They continuously make consumption and portfolio allocation decisions. Specifically, at any time t and state s, investor i has some financial wealth denoted by $a_{t,s}^i$. She chooses her consumption rate, $c_{t,s}^i$, and the fraction of her wealth to allocate to the market portfolio, $\omega_{t,s}^i$. The residual fraction, $1 - \omega_{t,s}^i$, is invested in the risk-free asset.

Note that $\omega_{t,s}^i$ also captures the investors' *leverage ratio*. We impose a leverage limit in the boom state $s = 1$:

$$\omega_{t,1}^i \leq \overline{\omega}_{t,1}, \tag{6}$$

where we require $\overline{\omega}_{t,1} \geq 1$ (to ensure market clearing). We allow for $\overline{\omega}_{t,1} = \infty$, in which case the leverage limit never binds. Our main result applies when the leverage limit may bind.

Remark 3 (Broadening the interpretation of the leverage limit). *We view the leverage limit as capturing a variety of unmodeled (and relevant) features that would make high-valuation investors' leverage ratio (to some extent) exogenous to the risks that we explicitly model. First, the leverage limit can capture a government-imposed leverage constraint. Second, the limit can capture a market-imposed leverage constraint due to unmodeled financial frictions such as moral hazard, adverse selection, lenders' uncertainty or their desire for safety. Third, the limit can also can be self-imposed: specifically, it can capture high-valuation investors' leverage choice in a richer environment that features unmodeled risks, e.g., diffusion risk in addition to the jump*

risk. In such an environment, as long as financial markets are incomplete, investors' leverage choice would be determined by a combination of unmodeled and modeled risks—and therefore would respond relatively less to changes in modeled risks.

For analytical tractability, we assume investors have log utility. The investors' problem (at time t and state s) can then be written as

$$V_{t,s}^i \left(a_{t,s}^i \right) = \max_{\left[c_{\tilde{t},\tilde{s}}, \omega_{t,\tilde{s}}^i \right]_{\tilde{t} \geq t, \tilde{s}}} E_{t,s}^i \left[\int_t^\infty e^{-\rho \tilde{t}} \log c_{\tilde{t}, \tilde{s}}^i d\tilde{t} \right] \tag{7}$$

$$\text{s.t.} \quad \begin{cases} da_{t,s}^i = \left(a_{t,s}^i \left(r_{t,s}^f + \omega_{t,s}^i \left(r_{t,s} - r_{t,s}^f \right) \right) - c_{t,s} \right) dt & \text{absent transition,} \\ a_{t,s'}^i = a_{t,s}^i \left(1 + \omega_{t,s}^i \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \right) & \text{if there is a transition to state } s' \neq s \end{cases} \tag{8}$$

and $\omega_{t,1}^i \leq \overline{\omega}_{t,1}$.

Here, $E_{t,s}^i [\cdot]$ denotes the expectation operator corresponding to investor i's beliefs for state transition probabilities.

Equilibrium in asset markets. Asset markets clear when the total wealth held by investors is equal to the value of the market portfolio both before and after the portfolio allocation decisions:

$$a_{t,s}^o + a_{t,s}^p = \omega_{t,s}^o a_{t,s}^o + \omega_{t,s}^p a_{t,s}^p = Q_{t,s} k_{t,s}. \tag{9}$$

When the conditions in (9) are satisfied, the market clearing condition for the risk-free asset (which is in zero net supply) holds.

Nominal rigidities and equilibrium in goods markets. The supply side of our model features nominal rigidities similar to the New Keynesian model. There is a continuum of competitive production firms that own the capital stock and produce the final good. For simplicity, these production firms have pre-set nominal prices that never change. Firms choose their capital utilization rate, $\eta_{t,s}$, to maximize their market value subject to demand constraints. They take into account that greater $\eta_{t,s}$ increases production according to Eq. (3) and that it leads to faster capital depreciation according to Eq. (4).

First consider the benchmark case without price rigidities. In this case, firms solve the problem:

$$\max_{\eta_{t,s}} \eta_{t,s} A k_{t,s} - \delta \left(\eta_{t,s} \right) Q_{t,s} k_{t,s}. \tag{10}$$

The optimality condition is given by

$$\delta' \left(\eta_{t,s} \right) Q_{t,s} = A. \tag{11}$$

That is, the frictionless level of utilization ensures that the marginal depreciation rate is equal to the marginal product of capital.

Next consider the case with price rigidities. In this case, firms solve problem (10) with the additional constraint that their output is determined by aggregate demand. As in the New Keynesian model, firms optimally meet this demand as long as their price exceeds their marginal cost. In a symmetric environment, the real price per unit of consumption good is one for all firms, and each firm's marginal cost is given by $\frac{\delta'(\eta_{t,s})Q_{t,s}}{A}$. Therefore, firms' optimality condition can be written as[4]

$$y_{t,s} = \eta_{t,s} A k_{t,s} = c^o_{t,s} + c^p_{t,s} \qquad \text{as long as } \delta'(\eta_{t,s}) Q_{t,s} \leq A. \tag{12}$$

Moreover, all output accrues to production firms in the form of earnings. Hence, the market portfolio can be thought of as a claim on all production firms.

Interest rate rigidity and monetary policy. Our assumption that production firms do not change their prices implies that the aggregate nominal price level is fixed. The real risk-free interest rate, then, is equal to the nominal risk-free interest rate, which is determined by the monetary authority's interest rate policy. We assume there is a lower bound on the nominal interest rate, which we set as zero for convenience: $r^f_{t,s} \geq 0$.

We model monetary policy as a sequence of interest rates, $\left\{ r^f_{t,s} \right\}_{t,s}$, and implied levels of factor utilization and asset price levels, $\left\{ \eta_{t,s}, Q_{t,s} \right\}_{t,s}$, chosen subject to the zero lower bound constraint. Absent price rigidities, factor utilization and asset price levels satisfy condition (11). Therefore, we define the *conventional output-stabilization policy* as

$$r^f_{t,s} = \max\left(0, r^{f*}_{t,s}\right) \text{ for each } s, \tag{13}$$

where $r^{f*}_{t,s}$ ("rstar") is recursively defined as the instantaneous interest rate that obtains when condition (11) holds and the planner follows the output-stabilization policy in (13) at all future times and states.

Our goal is to understand whether the planner might want to use monetary policy for prudential purposes in the boom state. In particular, we assume the planner follows the conventional output-stabilization policy in (13) for the recession and the recovery states $s \in \{2, 3\}$, but she might deviate from this rule in the boom state $s = 1$. For now, we allow the planner to choose an arbitrary path, $\left\{ r^f_{t,1}, Q_{t,1}, \eta_{t,1} \right\}_t$, that is consistent with the equilibrium conditions. We specify the monetary policy further in Section 5 and define the equilibrium below.

Definition 1. *The equilibrium is a collection of processes for allocations, prices, and returns such that capital evolves according to Eq. (4), its instantaneous return is given by Eq. (5), investors maximize their expected utility subject to a leverage limit in the boom state (cf. problem 7), asset markets clear (cf. Eq. (9)), goods markets clear (cf. Eq. (12)), and the monetary*

---

[4]If instead the marginal cost exceeded the price, $\frac{\delta'(0)Q_{t,s}}{A} > 1$, then these firms would choose $\eta_{t,s} = 0$ and produce $y_{t,s} = 0$. This case does not emerge in equilibrium.

authority follows the conventional output-stabilization policy in states $s \in \{2, 3\}$ [cf. Eq. (13)] and chooses a feasible path $\left\{ r_{t,1}^f, Q_{t,1}, \eta_{t,1} \right\}_t$ in state $s = 1$.

We next provide a generally applicable partial characterization of the equilibrium. In subsequent sections, we use this characterization to describe the equilibrium in the different states and policy regimes.

2.1. Equilibrium in the goods market

We start by establishing the equilibrium conditions in the goods market. In view of log utility, the investor's consumption is a constant fraction of her wealth, regardless of her portfolio choice:

$$c_{t,s}^i = \rho a_{t,s}^i. \tag{14}$$

This leads to a tight relationship between output and asset prices. Combining Eqs. (14) and (9) implies that aggregate consumption is a constant fraction of aggregate wealth,

$$c_{t,s}^o + c_{t,s}^p = \rho Q_{t,s} k_{t,s}.$$

Combining this result with the goods market clearing condition in Eq. (12), we obtain *the output-asset price relation*,

$$y_{t,s} = A \eta_{t,s} k_{t,s} = \rho Q_{t,s} k_{t,s}. \tag{15}$$

Intuitively, greater asset prices increase aggregate demand, output, and factor utilization. Combining Eqs. (11) and (15), we find that the efficient level of output utilization solves

$$\delta' (\eta^*) \eta^* = \rho. \tag{16}$$

Note that optimal capital utilization is the same across all states. We assume the following regularity conditions on the depreciation function to ensure that there exists a unique solution to Eq. (16):

Assumption 2. $\delta(\eta)$ is strictly increasing and convex over $\mathbb{R}_+$ with $\delta'(0) < \rho$ and $\lim_{\eta \to \infty} \delta'(\eta) \geq \rho$.

Combining Eqs. (15) and (16), we find that there is an efficient asset price level:

$$Q^* = \frac{A \eta^*}{\rho}. \tag{17}$$

This is the level of asset prices such that the associated aggregate demand leads to efficient capital utilization (and ensures that actual output is exactly at potential output). When $Q_{t,s} < Q^*$, we have $\eta_{t,s} < \eta^*$: capital is utilized below its efficient level, which we interpret as a demand recession. Note also that, using the one-to-one relationship between factor utilization and asset

prices in (15), we have $\frac{\eta_{t,s}}{\eta^*} = \frac{Q_{t,s}}{Q^*}$: the degree of underutilization relative to the efficient level is proportional to the ratio of the asset price level to the efficient asset price level.

Next note that note that Eq. (15) implies the equilibrium dividend yield is given by $\frac{y_{t,s}}{Q_{t,s}k_{t,s}} = \rho$. Substituting this into Eq. (5), and using the relationship between $\eta_{t,s}$ and $Q_{t,s}$, the equilibrium return on the market portfolio is given by:

$$r_{t,s} = \rho + g_s - \delta \left(\frac{Q_{t,s}}{Q^*} \eta^* \right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}}. \tag{18}$$

2.2. Equilibrium in asset markets

We next establish the equilibrium conditions in asset markets. For these markets, the key state variable is investors' relative wealth shares, which we define as

$$\alpha_{t,s}^i \equiv \frac{a_{t,s}^i}{Q_{t,s}k_{t,s}} \text{ for } i \in \{o, p\}. \tag{19}$$

Note that investors' wealth shares sum to one, $\alpha_{t,s}^o + \alpha_{t,s}^p = 1$ [cf. Eq. (9)].

In the appendix, we characterize investors' wealth share after a transition in terms of their leverage ratio

$$\frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i} - 1 = \left(\omega_{t,s}^i - 1 \right) \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{20}$$

When the transition increases the asset price, $Q_{t,s'} > Q_{t,s}$, an investor's wealth share increases after the transition, $\alpha_{t,s'}^i > \alpha_{t,s}^i$, if and only if she has above-average leverage, $\omega_{t,s}^i > 1$. The converse happens if the transition decreases the asset price.

Note also that Eq. (20) establishes a one-to-one relationship between $\alpha_{t,s'}^i$ and $\omega_{t,s}^i$ (as long as $Q_{t,s'} \neq Q_{t,s}$, which is the case in our model). Hence, we can think of the investor as choosing her wealth share after transition, $\alpha_{t,s'}^i$, and adjusting her leverage ratio to obtain this outcome. Thus, we can state the investor's portfolio optimality condition as

$$r_{t,s} - r_{t,s}^f + \lambda_s^i \frac{\alpha_{t,s}^i}{\alpha_{t,s'}^i} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \geq 0, \tag{21}$$

with equality when the leverage limit doesn't bind (see Appendix A.1 for a derivation). As long as the investor is unconstrained, she invests in the market portfolio until the risk-adjusted expected excess return is zero. The risk-adjusted return captures aggregate price changes $\left(\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right)$ as well as the adjustment of marginal utility relative to other investors if there is a transition $\left(\frac{\alpha_{t,s}^i}{\alpha_{t,s'}^i} \right)$. For the equilibria we analyze, the leverage limit never binds for pessimists. Consequently, the optimality condition (21) always holds as equality for pessimists but it might apply as inequality for optimists.

Finally, combining Eqs. (9), (19) and (20), we can see that asset markets clear as long as

14

investors' wealth shares after transition, $\left\{\alpha_{t,s'}^i\right\}_{i\in\{o,p\}}$, sum to one. Therefore, the equilibrium in asset markets reduces to finding wealth shares that solve (21) for each type and that satisfy $\alpha_{t,s'}^o + \alpha_{t,s'}^p = 1$.

Next consider the evolution of investors' wealth shares if there is no state transition. In Appendix A.1.2, we show that

$$\frac{\dot\alpha_{t,s}^i}{\alpha_{t,s}^i} = \lambda_s^p \frac{\alpha_{t,s}^p}{\alpha_{t,s'}^p}\left(1 - \frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i}\right). \tag{22}$$

Pessimists' beliefs (superscript p) appear in this expression because the optimality condition (21) always holds as equality for them, so we can use their beliefs to price assets. This expression illustrates that investors face a trade-off across states. If an investor chooses $\alpha_{t,s'}^i > \alpha_{t,s}^i$ (resp. $\alpha_{t,s'}^i < \alpha_{t,s}^i$) so that her wealth share increases (resp. decreases) after a state transition, then she also has $\dot\alpha_{t,s}^i < 0$ (resp. $\dot\alpha_{t,s}^i > 0$) so her wealth share shrinks (resp. grows) if there is no state transition.

Special case with non-binding leverage limits. When the leverage limit doesn't bind for optimists, these equations can be simplified further. In particular, Eq. (21) holds as equality for both types of investors, which implies $\lambda_s^o \frac{\alpha_{t,s}^o}{\alpha_{t,s'}^o} = \lambda_s^p \frac{\alpha_{t,s}^p}{\alpha_{t,s'}^p}$. Combining this equality with the market clearing condition (9), we obtain a closed-form solution:

$$\frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i} = \frac{\lambda_s^i}{\overline\lambda_{t,s}} \text{ where } \overline\lambda_{t,s} = \alpha_{t,s}\lambda_{t,s}^o + \left(1 - \alpha_{t,s}\right)\lambda_{t,s}^o. \tag{23}$$

Here $\overline\lambda_{t,s}$ denotes the wealth-weighted average of the transition probability. After substituting this expression into Eq. (22), we solve for investors' wealth dynamics as:

$$\frac{\dot\alpha_{t,s}^i}{\alpha_{t,s}^i} = -\left(\lambda_s^i - \overline\lambda_{t,s}\right). \tag{24}$$

These expressions are intuitive. When type i investors assign an above-average probability to transition, $\lambda_s^i > \overline\lambda_{t,s}$, their wealth share increases after a transition but drifts downward absent a transition. Conversely, when investors assign a below-average transition probability, their wealth share declines after a transition but drifts upward absent a transition.

Remark 4 (Role of market incompleteness due to binding leverage limits). *Eqs. $(21-24)$ clarify the difference of this model with the one in Caballero and Simsek (2017). Specifically, Eqs. (23) and (24) are the same as their counterparts in Caballero and Simsek (2017), where we allow investors to trade transition risks via Arrow-Debreu securities. The intuition is that, as long as the leverage limit does not bind, the market portfolio and the risk-free asset are sufficient to dynamically complete the market. The main difference in this setting is that the leverage limit*

can *bind, in which case the wealth-share dynamics are different than in Caballero and Simsek (2017) and are characterized by Eqs.* (21) *and* (22).

3.   The recession and aggregate demand externalities

We next characterize the equilibrium in the recession state (as well as in the recovery state). We also illustrate the aggregate demand externalities that motivate policy intervention. Since our focus is on the boom state, we relegate the details to Appendix A.2 and state the key equations and the results relevant for our analysis. For the rest of the paper, with a slight abuse of notation, we often drop the superscript o from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha_{t,s}^o.$$

Pessimists' wealth share is the complement of this expression, $\alpha_{t,s}^p = 1 - \alpha_{t,s}$. We will describe the remaining equilibrium variables as functions of optimists' wealth share, so this convention will considerably simplify the notation.

Under appropriate parametric restrictions (Assumption A1) we show that the recovery state $s = 3$ features positive interest rates, efficient asset prices, and efficient factor utilization, $r_{t,3}^f > 0, Q_{t,3} = Q^*, \eta_{t,3} = \eta^*$, whereas the recession state $s = 2$ features zero interest rates, inefficiently low asset prices, and inefficient factor utilization, $r_{t,2}^f = 0, Q_{t,2} < Q^*, \eta_{t,2} < \eta^*$. The equilibrium in the recovery state is straightforward since there is no further transition and no speculation. We then proceed backwards, starting with a description of the equilibrium in the recession state.

Equilibrium in the recession.   Since there is no leverage limit in this state, Eq. (21) holds as equality for both types of investors. We aggregate this expression across investors (using Eq. (23)), and substitute for $r_{t,2}$ from Eq. (18) and $Q_{t,3} = Q^*$, to obtain:

$$\rho + g_2 - \delta \left(\frac{Q_{t,2}}{Q^*} \eta^* \right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2} \left(1 - \frac{Q_{t,2}}{Q^*} \right) = r_{t,2}^f. \tag{25}$$

We refer to this expression as the *risk balance condition*: it says that the equilibrium risk-adjusted return on the market portfolio (evaluated with the wealth-weighted average belief) is equal to the risk-free interest rate.

As a preliminary step, consider the outcomes that would obtain if the interest rate were unconstrained. In this case, substituting $Q_{t,2} = Q^*$ into the risk balance condition (25), we obtain an expression for the output-stabilizing interest rate: $r_{t,2}^{f,*} = \rho + g_2 - \delta(\eta^*)$. For intuition, consider the effect of lowering g_2. This exerts downward pressure on asset prices due to low expected growth in output and earnings. Monetary policy responds by lowering the risk-free interest rate, $r_{t,2}^{f,*}$, and keeps asset prices at the efficient level, $Q_{t,2} = Q^*$. By lowering the risk-free rate, monetary policy ensures that investors continue to hold the market portfolio at

the efficient asset price level, even though they expect low output growth.

We assume g_2 is sufficiently low so that the implied output-stabilizing interest rate violates the lower bound, $r_{t,2}^{f,*} < 0$. Consider the outcomes with a binding interest rate lower bound. Substituting $r_{t,2}^f = 0$ into the risk balance condition (25), we obtain an expression that characterizes the asset price, $Q_{t,2}$. Intuitively, the only way condition (25) can be satisfied when $r_{t,2}^f$ cannot decline below zero is for $Q_{t,2}$ to fall below below Q^*. This asset price decline increases the return of the market portfolio, which in turn ensures that investors continue to hold the market portfolio despite lower expected output growth. However, the decline in $Q_{t,2}$ also lowers aggregate spending and triggers a demand recession.

Importantly, Eq. (25) suggests that, when the wealth-weighted belief is more optimistic (greater $\overline{\lambda}_{t,2}$), a smaller decline in $Q_{t,2}$ is sufficient to reestablish the risk balance condition. We verify this intuition in the appendix. Formally, we characterize the asset price and optimists' wealth share, $(Q_{t,2}, \alpha_{t,2})$, as the solution to a differential equation in the time domain (see Eq. (A.9)). We write the solution as $Q_{t,2} = Q_2(\alpha_{t,2})$ for each $\alpha_{t,2} \in [0,1]$. We show that the function, $Q_2(\cdot)$, satisfies

$$Q_2(\alpha) < Q^* \text{ and } \frac{dQ_2(\alpha)}{d\alpha} > 0 \text{ for each } \alpha \in (0,1). \tag{26}$$

In particular, a greater wealth-share for optimists increases the asset price and brings it closer to the frictionless level.

Recall from Eq. (15) that there is a one-to-one relationship between asset prices and factor utilization. Hence, Eq. (26) implies $\eta_{t,2} < \eta^*$: the recession features an inefficiently low level of capital utilization. We capture the welfare costs of underutilization with the concept of a *gap value function,* which we first introduced in Caballero and Simsek (2017).

Gap value function. To define the gap value function, let b denote a superscript representing beliefs about transition probabilities. The planner can have different beliefs from optimists and pessimists, so b takes one of three values $\{o, p, pl\}$. For a fixed b, we use $V_{t,s}^{i,b}(a_{t,s}^i)$ to denote type i investors' equilibrium value calculated according to type b beliefs. In view of log utility, the value function takes the form

$$V_{t,s}^{i,b}(a_{t,s}^i) = \frac{\log(a_{t,s}^i/Q_{t,s})}{\rho} + v_{t,s}^{i,b}.$$

The normalized value function $v_{t,s}^{i,b}$ captures the value when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^i = Q_{t,s}$). We further decompose this term as follows:

$$v_{t,s}^{i,b} = v_{t,s}^{i*,b} + w_{t,s}^b. \tag{27}$$

The *frictionless value function* $v_{t,s}^{i*,b}$ is the value that obtains in a counterfactual economy where the evolution of wealth shares are left unchanged but asset prices are equal to the frictionless level, $Q_{t,s} = Q^*$ for each t, s. This captures all determinants of welfare (including the benefits/costs from speculation) except for suboptimal factor utilization. The residual term, $w_{t,s}^b$, corresponds to the *gap value function*. This term captures the welfare losses due to suboptimal factor utilization evaluated according to investors' preferences (and type b beliefs).

In the appendix, we formalize this intuition by establishing that the gap value function solves the following differential equation:

$$\rho w_{t,s}^b - \frac{\partial w_{t,s}^b}{\partial t} = W(Q_{t,s}) + \lambda_s^b \left(w_{t,s'}^b - w_{t,s}^b \right), \tag{28}$$
$$\text{where } W(Q_{t,s}) \equiv \log \frac{Q_{t,s}}{Q^*} - \frac{1}{\rho} \left(\delta \left(\frac{Q_{t,s}}{Q^*} \eta^* \right) - \delta(\eta^*) \right).$$

The function $W(Q_{t,s})$ is strictly concave with a maximum at $Q_{t,s} = Q^*$ and maximum value equal to zero, $W(Q^*) = 0$ (cf. Eq. (16)). $W(Q_{t,s}) \leq 0$ captures the instantaneous losses in welfare when the asset price (and therefore factor utilization) deviates from its efficient level, $Q_{t,s} \neq Q^*$. Therefore, the gap value $w_{t,s}^b$ corresponds to the present discounted value of expected welfare losses due to price rigidities and inefficient factor utilization.

In our welfare analysis, we mostly focus on the gap value function calculated according to the planner's belief, $b = pl$. This sidesteps questions about whether speculation increases or reduces welfare (see Brunnermeier et al. (2014) for further discussion). Our analysis aligns with the mandates of monetary policy in practice: the planner in our model exclusively focuses on minimizing output gaps relative to a frictionless benchmark (similar to Kocherlakota (2014) and Stein (2014)).[5] Following Brunnermeier et al. (2014), we assume the planner's beliefs are in the convex hull of optimists' and pessimists' beliefs: $\lambda_1^{pl} \in [\lambda_1^o, \lambda_1^p]$ and $\lambda_2^{pl} \in [\lambda_2^p, \lambda_2^o]$. Our results are qualitatively robust to the choice of planner's beliefs in these sets.

Gap value in the recession: Aggregate demand externalities. In the appendix, we show that the planner's gap value function in the recession can be written as $w_{t,2}^{pl} = w_2^{pl}(\alpha_{t,2})$, where $w_2^{pl}(\cdot)$ is a function that satisfies:

$$w_2^{pl}(\alpha) < 0 \text{ and } \frac{dw_2^{pl}(\alpha)}{d\alpha} > 0 \text{ for each } \alpha \in (0,1). \tag{29}$$

As expected, the gap value is strictly negative. Moreover, a greater wealth-share for optimists shrinks the gap value. The welfare gap is smaller when optimists have more wealth, since optimists' wealth increases asset prices and aggregate demand and mitigates the underutilization

---

[5] In Caballero and Simsek (2017), we illustrate that (under appropriate parametric restrictions) macroprudential policy that restricts investors' risk taking can generate a *Pareto* improvement in welfare. That is, the planner can make everyone better off even if she focuses on the *total value* (not just the gap value) and evaluates each investor's expected value *according to her own belief*.

of capital [cf. Eqs. (26) and (28)].

Note that optimists' wealth share is an *endogenous* and *aggregate* state variable that depends on the amount of financial speculation that takes place in the boom state. In particular, the positive relationship between optimists' wealth and the gap value in (29) illustrates *aggregate demand externalities* that motivate policy interventions during the boom. Individual optimists that take on leverage during the boom (and pessimists that lend to them) do not internalize the effects of their financial decisions on asset prices in the recession. In subsequent sections, we investigate whether prudential policies can help correct these externalities.

4. The boom: benchmark without prudential monetary policy

We now turn to our main focus: the equilibrium in the boom state. In this section, we analyze the benchmark case without PMP, that is, when monetary policy follows the conventional output-stabilization policy in (13) in state $s = 1$. We use this setup to illustrate that macroprudential policy that tightens the leverage limit can internalize the aggregate demand externalities. In the next section we introduce PMP and show that it can accomplish similar financial stability objectives to macroprudential policy.

Recall that investors face a (possibly time-varying) leverage limit, $\omega^i_{t,1} \leq \overline{\omega}_{t,1}$. We assume the leverage limit can be written as a function of optimists' wealth share, $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$. This assumption ensures that $\alpha_{t,1}$ is the only state variable. We denote the equilibrium variables as functions of optimists' wealth share and the leverage limit *function*: $\alpha_{t,2} = \alpha_2(\alpha, \overline{\omega}_1(\cdot))$ denotes optimists' wealth share *after* transition when their current wealth share is $\alpha_{t,1} = \alpha$ and the leverage limit is described by $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$ for each t. We use the notation $\alpha_2(\alpha, \infty)$ to denote the equilibrium when there is no leverage limit: $\overline{\omega}_1(\alpha) = \infty$ for each α.

Under appropriate parametric restrictions (Assumptions A2-A3 in the appendix) we show that the boom without PMP features positive interest rates, efficient asset prices, and efficient factor utilization, $r^f_{t,1} > 0, Q_{t,1} = Q^*, \eta_{t,1} = \eta^*$. To characterize this equilibrium, consider the intermediate cases, $\alpha_{t,1} \in (0,1)$ (the corner cases are straightforward and relegated to the appendix). The leverage limit doesn't bind for pessimists but it might bind for optimists. Using Eq. (21) for pessimists, and substituting $r_{t,1}$ from Eq. (18) and $Q_{t,1} = Q^*, Q_{t,2} = Q_2(\alpha_{t,2})$, we obtain

$$r^f_1(\alpha, \overline{\omega}_1) = \rho + g_1 - \delta(\eta^*) - \lambda^p_1 \frac{1-\alpha}{1 - \alpha_2(\alpha, \overline{\omega}_1)} \left(\frac{Q^*}{Q_2(\alpha_2(\alpha, \overline{\omega}_1))} - 1 \right). \tag{30}$$

This is the risk balance condition according to pessimists (cf. Eq. (25)). The condition characterizes the output-stabilizing interest rate given investors' wealth shares. Assumption A2 ensures that $r^f_1(\alpha, \overline{\omega}_1) > 0$ when $\alpha = 0$, that is, the interest rate is above the lower bound if pessimists dominate.

Hence, it remains to characterize the function $\alpha_2(\alpha, \overline{\omega}_1)$. First consider the special case without a leverage limit, $\overline{\omega}_1 = \infty$ for each α. In this case, Eq. (23) provides a closed-form

solution:

$$\alpha_2\left(\alpha,\infty\right) = \alpha\frac{\lambda_1^o}{\overline{\lambda}_1\left(\alpha\right)} < \alpha. \tag{31}$$

Recall that we use the notation $\alpha_2\left(\alpha,\infty\right)$ to denote optimists' equilibrium wealth share without a leverage limit. The expression $\overline{\lambda}_1\left(\alpha\right) \equiv \alpha\lambda_1^o + \left(1-\alpha\right)\lambda_1^p$ denotes the wealth-weighted average probability as a function of optimists' wealth share. Using Eq. (20), we can solve for the corresponding leverage ratio in closed form:

$$\omega_1^o\left(\alpha,\infty\right) = 1 + \frac{1 - \frac{\lambda_1^o}{\overline{\lambda}_1(\alpha)}}{\frac{Q^*}{Q_2\left(\frac{\alpha\lambda_1^o}{\overline{\lambda}_1(\alpha)}\right)} - 1} > 1. \tag{32}$$

Optimists have above-average leverage during the boom, which induces a decline in their wealth share after transition to the recession.

Next consider the case with a leverage limit. Suppose $\overline{\omega}_1\left(\alpha\right) \leq \omega_1^o\left(\alpha,\infty\right)$ so that the limit binds (the other case is the same as before). Then, optimists' leverage ratio is determined by the limit:

$$\omega_1^o\left(\alpha,\overline{\omega}_1\right) = \overline{\omega}_1\left(\alpha\right). \tag{33}$$

To find optimists' wealth share after transition, we consider Eq. (20) for the boom state $s = 1$:

$$\begin{aligned}
\frac{\alpha_2\left(\alpha,\overline{\omega}_1\right)}{\alpha} &= 1 - \left(\overline{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{Q_1}{Q_2} - 1\right], \\
\text{where } Q_1 &= Q^* \text{ and } Q_2 = Q_2\left(\alpha_2\left(\alpha,\overline{\omega}_1\right)\right).
\end{aligned} \tag{34}$$

The first line of this expression is the microfounded version of Eq. (2) from the introduction. The second line substitutes the equilibrium prices for the boom and the recession states. The last equation is the microfounded version of Eq. (1). As illustrated by Figure 1, the equilibrium can be visualized as the intersection of two increasing relations. In Appendix A.3, we show that under appropriate regularity conditions (Assumption A3), Eq. (34) has a unique solution that satisfies $\alpha_2\left(\alpha,\overline{\omega}_1\right) \in \left[\alpha_2\left(\alpha,\infty\right),\alpha\right]$.

Finally, applying Eq. (22), we obtain the dynamics of optimists' wealth share absent a transition as

$$\frac{\dot{\alpha}_{t,1}}{\alpha_{t,1}} = \lambda_1^p \frac{1 - \alpha_{t,1}}{1 - \alpha_2\left(\alpha_{t,1},\overline{\omega}_1\right)}\left(1 - \frac{\alpha_2\left(\alpha_{t,1},\overline{\omega}_1\right)}{\alpha_{t,1}}\right) \leq \left(1 - \alpha_{t,1}\right)\left(\lambda_1^p - \lambda_1^o\right). \tag{35}$$

The weak inequality is satisfied as equality when the leverage limit doesn't bind (i.e., when α_2 is given by Eq. (31)). It is also easy to see that $\dot{\alpha}_{t,1}/\alpha_{t,1}$ is a decreasing function of α_2: if optimists obtain a greater wealth share after transition to recession, then their wealth share grows more slowly if there is no transition.

To summarize the equilibrium without PMP, the asset price during the boom is at its efficient level, $Q_1(\alpha, \overline{\omega}_1) = Q^*$, and the equilibrium interest rate is given by (30). If optimists' leverage is unconstrained, their wealth share after transition and their leverage ratio are given by Eqs. (31) and (32). If their leverage ratio is constrained, these values are given by Eqs. (33) and (34). Optimists' wealth share evolves according to (35).

Our next result describes how macroprudential policy that tightens the leverage limit affects this equilibrium. This provides a useful benchmark for the next section where we assume macroprudential policy is imperfect and investigate whether PMP can provide similar financial stability benefits.

Proposition 1. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the benchmark equilibrium without PMP, $Q_1(\cdot) = Q^*$. Fix a level $\alpha \in (0,1)$ that is associated with some binding leverage limit, $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha, \infty)$. Decreasing the leverage limit increases optimists' wealth share after a transition to recession: $\frac{d\alpha_2(\alpha, \overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$. It also slows down the growth rate of optimists' wealth share if the boom persists, $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d\overline{\omega}_1(\alpha)} < 0$.*

For a sketch proof (completed in Appendix A.3), note that optimists' wealth decline after transition is increasing in their leverage ratio, $\omega_1 - 1$ [cf. Eq. (34)]. Tightening the leverage limit reduces optimists' leverage ratio, $\tilde{\omega}_1 - 1 < \omega_1 - 1$, which in turn mitigates their wealth decline. This increases the price level in the recession, Q_2, which further boost optimists' wealth. In equilibrium, optimists' wealth share and the asset price in the recession settle at a higher level, $\alpha_2(\alpha, \tilde{\omega}_1) > \alpha_2(\alpha, \overline{\omega}_1)$ and $Q_2(\alpha_2(\alpha, \tilde{\omega}_1)) > Q_2(\alpha_2(\alpha, \overline{\omega}_1))$. The left panel of Figure 1 (in the introduction) illustrates the virtuous cycle that results from tightening the leverage limit.

Recall that increasing optimists' wealth share in the recession internalizes aggregate demand externalities [cf. Eq. (29)]. Therefore, Proposition 1 illustrates how macroprudential policy that tightens the leverage limit *can* improve welfare. At the same time, the welfare effects do not follow immediately because tightening the leverage limit also slows down the growth of optimists' wealth share if the recession is not realized, as illustrated by the last part of Proposition 1. In a dynamic setting, optimists' wealth share can also be useful in future recessions and thus macroprudential policy involves a trade-off. We investigate this trade-off in Caballero and Simsek (2017), where we show that the benefits from an immediate transition to recession often dominate the costs from worsening future recessions (in view of discounting). In particular, we show that (under regularity conditions and starting from a no-policy benchmark) adopting some macroprudential policy improves welfare.

5. Prudential monetary policy

We now assume that macroprudential policy is inflexible: the planner cannot change the existing leverage constraints. Instead, we introduce our main ingredient and allow monetary policy in the boom state to be used for prudential purposes. We start by establishing a negative result:

when there is no leverage limit, PMP is useless because optimists endogenously change their risk taking to undo the prudential benefits. We then consider the case with a leverage limit and establish that, when there is *some* leverage limit, monetary policy can replicate the prudential effects of tightening this limit. Specifically, our main result establishes that, up to a first order, the welfare effects of PMP are the same as the effects of directly tightening the leverage limit.

Formally, suppose that in the boom state the planner does not follow the rule in (13) but instead sets the interest rate to target an asset price level, $Q_{t,1}$, which might be lower than the efficient level, $Q_{t,1} \leq Q^*$. We assume the planner's price target can be written as a function of optimists' wealth share:

$$Q_{t,1} = Q_1(\alpha_{t,1}) \leq Q^*.$$

We denote the equilibrium variables as functions of the PMP *function* (in addition to the earlier variables): $\alpha_2(\alpha, \overline{\omega}_1(\cdot), Q_1(\cdot))$ denotes optimists' wealth share after transition, when monetary policy is described by $Q_{t,1} = Q_1(\alpha_{t,1})$ for each t. We use the same notation as in the previous section to denote the equilibrium in the benchmark in which the planner follows the conventional output-stabilization policy: e.g., $\alpha_2(\alpha, \overline{\omega}_1)$ denotes the equilibrium when monetary policy is described by $Q_{t,1} = Q^*$ for each t.

5.1.   No leverage limit

First consider the case without a leverage limit, $\overline{\omega}_1 = \infty$. In this case, we establish a *negative* result: PMP can only worsen the gap value (i.e., reduce welfare).

Proposition 2. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the case without a leverage limit, $\overline{\omega}_1 = \infty$, and some PMP, $Q_1(\cdot)$. Optimists' wealth share after transition and the evolution of their wealth share are the same as in the benchmark without prudential policy (in particular, Eq. (31) holds). The policy lowers the planner's gap value relative to the benchmark with conventional output-stabilization policy:*

$$w_1^{pl}(\alpha, \infty, Q_1) \leq w_1^{pl}(\alpha, \infty).$$

The first part of Proposition 2 says that PMP, by itself, does not affect the evolution of investors' wealth shares. The second part follows as a corollary. Since the policy does not affect wealth shares, it only affects the gap value through its impact on the asset price during the boom, $Q_{t,1}$ [cf. Eq. (28)]. Lowering $Q_{t,1}$ below Q^* makes factor utilization less efficient and decreases welfare: $W(Q_{t,1}) < W(Q^*)$ when $Q_{t,1} < Q^*$. Put differently, the policy has no benefits, but it has some costs due to low asset prices and inefficient factor utilization in the boom state.

The key step to our argument is that the policy does *not* affect optimists' wealth share after transition, $\alpha_2(\alpha, \infty, Q_1) = \alpha_2(\alpha, \infty) = \alpha \frac{\lambda_1^o}{\lambda_1(\alpha)}$ [cf. Eq. (31)]. To understand this feature, consider the equilibrium for an intermediate case, $\alpha \in (0, 1)$, and note that the policy affects

optimists' equilibrium leverage ratio. In particular, we have the following version of Eq. (34):

$$\frac{\alpha_2\left(\alpha,\infty\right)}{\alpha} = 1 - \left(\omega_1^o\left(\alpha,\infty,Q_1\right) - 1\right)\left[\frac{Q_1\left(\alpha\right)}{Q_2\left(\alpha_2\left(\alpha,\infty\right)\right)} - 1\right].$$

Note that a decline in $Q_1\left(\alpha\right)$ *does* result in a smaller price drop after transition (the term inside the brackets). Therefore, the policy leaves optimists' wealth share after transition (α_2) unchanged because it induces optimists to *increase* their leverage ratio, $\omega_1^o\left(\alpha,\infty,Q_1\right) > \omega_1^o\left(\alpha,\infty\right)$. Put differently, the prudential effects of the policy are neutralized by an increase in optimists' risk taking. Optimists increase their leverage because they perceive the transition to recession as less risky due to a smaller asset price drop after the transition.

5.2. With leverage limit

The previous discussion suggests that PMP *can* affect investors' equilibrium exposures if optimists are constrained by *some* leverage limit. Consider a situation in which there is a limit that binds for optimists so that $\omega_1^o\left(\alpha,\overline{\omega}_1,Q_1\right) = \overline{\omega}_1\left(\alpha\right)$. Then, we have the following version of Eq. (34):

$$\frac{\alpha_2\left(\alpha,\overline{\omega}_1,Q_1\right)}{\alpha} = 1 - \left(\overline{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{Q_1\left(\alpha\right)}{Q_2\left(\alpha_2\left(\alpha,\overline{\omega}_1,Q_1\right)\right)} - 1\right]. \tag{36}$$

In this case, since $\overline{\omega}_1\left(\alpha\right)$ is fixed, a decline in $Q_1\left(\alpha\right)$ translates into an increase in optimists' wealth share after transition. By reducing asset prices during the boom, the planner reduces the price drop after a transition to recession, which supports optimists' balance sheets. The following result formalizes this intuition and shows that monetary policy can replicate the prudential effects of tightening the leverage limit.

Proposition 3. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the benchmark equilibrium without PMP, $Q_1\left(\cdot\right) = Q^*$. Fix a level $\alpha \in \left(0,1\right)$ that is associated with some leverage limit, $\overline{\omega}_1\left(\alpha\right) < \infty$ (that might or might not bind). Consider an alternative leverage limit $\tilde{\omega}_1\left(\cdot\right)$ that agrees with $\overline{\omega}_1\left(\cdot\right)$ everywhere except for α and that satisfies $\tilde{\omega}_1\left(\alpha\right) < \min\left(\overline{\omega}_1\left(\alpha\right),\omega_1^o\left(\alpha,\infty\right)\right)$, and a PMP $\tilde{Q}_1\left(\cdot\right)$ that agrees with $Q_1\left(\cdot\right)$ everywhere except for α. Then:*

(i) There exists $\tilde{Q}_1\left(\alpha\right) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on optimists' wealth share after transition as the alternative leverage limit (without PMP):

$$\alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) = \alpha_2\left(\alpha,\tilde{\omega}_1\right).$$

Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial\tilde{Q}_1\left(\alpha\right)}{\partial\tilde{\omega}_1\left(\alpha\right)} > 0$.

(ii) PMP requires setting a higher interest rate than the benchmark without policy:

$$r_1^f\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f\left(\alpha,\overline{\omega}_1\right).$$

Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right)}{\partial\tilde{\omega}_1\left(\alpha\right)} < 0$.

The first part of Proposition 3 shows that monetary policy can replicate the prudential effects of tightening the leverage limit that we established in Proposition 1. For a sketch proof (completed in Appendix A.4), note that optimists' wealth decline after a transition depends on the product of their (above-average) leverage and the price decline, $(\omega_1 - 1)\left[\frac{Q_1}{Q_2} - 1\right]$ [cf. Eq. (36)]. Recall that tightening the leverage limit mitigates optimists' wealth decline by reducing their leverage ratio, $\tilde{\omega}_1 - 1 < \omega_1 - 1$. For a given asset price Q_2, monetary policy can achieve the same wealth decline for optimists at the leverage limit, $\omega_1 = \overline{\omega}_1$, by reducing the asset price decline, $\frac{\tilde{Q}_1}{Q_2} - 1 < \frac{Q^*}{Q_2} - 1$. This policy increases the price level in the recession, Q_2, which generates a similar virtuous cycle as a policy that directly tightens the leverage limit. The right panel of Figure 1 illustrates how PMP generates effects that are very similar to tightening the leverage limit.

In fact, the monetary authority can choose $\tilde{Q}_1$ so that optimists' wealth share and the equilibrium price in the recession settle *exactly* at the same level as if the regulator had tightened the leverage limit, $\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2\left(\alpha, \tilde{\omega}_1\right)$ and $Q_2\left(\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right)\right) = Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)$. Specifically, after substituting these expressions into Eq. (36), we characterize $\tilde{Q}_1$ as the unique solution to

$$\left(\overline{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{\tilde{Q}_1}{Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)} - 1\right] = \left(\tilde{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)} - 1\right]. \qquad (37)$$

Hence, $\tilde{Q}_1$ is the asset price that replicates optimists' wealth decline after accounting for the endogenous price adjustment in the recession.

The second part of Proposition 3 shows that PMP requires raising the interest rate above the conventional policy benchmark with output stabilization. As expected, targeting a lower asset price requires a higher interest rate. This result offers an alternative interpretation for how PMP works. Recall that, if there is an instantaneous transition to the recession, then the interest rate will decline to zero with or without PMP, $r_2^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = r_2^f\left(\alpha, \overline{\omega}_1\right) = 0$. Hence, by increasing the interest rate during the boom, PMP increases the size of the interest rate *cut* in case there is a transition to recession, $r_1^f - r_2^f$. For a given level of Q_2, this reduces the asset price *decline* after transition to recession, Q_1/Q_2. A smaller asset price decline supports optimists' wealth share after transition, α_2, and increases the asset price level Q_2 (which triggers the virtuous cycle described earlier). Thus, the policy can be thought of as increasing the interest rate to create room for an interest rate *cut* and mitigate the impact of negative asset price shocks in the future.

Proposition 3 is essentially static: it considers a policy change at a particular instant while leaving the policy at other times unchanged. This is useful for illustrating how PMP works, but it does not have an impact on the *dynamic* equilibrium. In addition, since PMP has costs as well as benefits, there is the remaining question of how it affects welfare. We next present our main result, which generalizes Proposition 3 to a dynamic setting and shows that the welfare

24

effects of prudential policy are also (locally) equivalent to tightening the leverage limit.

To state the result, we parameterize the leverage limit function, $\overline{\omega}(\alpha, l)$ where $l \in L \subset \mathbb{R}_+$, and lower levels of l correspond to a tighter leverage limit, $\frac{\partial \overline{\omega}(\alpha,l)}{\partial l} > 0$ for $\alpha \in (0,1)$. An example is the simple leverage limit function

$$\overline{\omega}_1(\alpha, l) = l \text{ with } l \in L = (1, \infty). \tag{38}$$

Here, the leverage limit doesn't depend on α and a lower l corresponds to a tighter limit for all α. Whenever we parameterize the leverage limit function, we simplify the notation by denoting the corresponding equilibrium variables with $\alpha_2(\alpha, l, Q_1)$ (as opposed to $\alpha_2(\alpha, \overline{\omega}_1(\cdot, l), Q_1)$).

Proposition 4. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the case with some leverage limit function, $\overline{\omega}_1(\alpha, l)$, parameterized so that lower levels of l correspond to a tighter limit.*

(i) For each $\tilde{l} < l$ in a sufficiently small neighborhood of l, there exists a PMP, denoted by $Q_1\left(\cdot, \tilde{l}\right)$, such that optimists' equilibrium wealth share after transition is the same as when the leverage limit is given by $\overline{\omega}_1\left(\alpha, \tilde{l}\right)$ without PMP:

$$\alpha_2\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) = \alpha_2\left(\alpha, \tilde{l}\right) \text{ for each } \alpha \in (0,1).$$

(ii) For small policy changes, the welfare effects of PMP are the same as the welfare effects of tightening the leverage limit directly:

$$\frac{dw_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right)}{d\tilde{l}}\Big|_{\tilde{l}=l} = \frac{dw_1^{pl}\left(\alpha, \tilde{l}\right)}{d\tilde{l}}\Big|_{\tilde{l}=l}. \tag{39}$$

The first part of Proposition 4 follows from a similar analysis as in Proposition 3. In particular, for each $\alpha \in (0,1)$, the price level $Q_1\left(\alpha, \tilde{l}\right) = \tilde{Q}_1$ corresponds to the policy that replicates the prudential effects of the tighter leverage limit, $\overline{\omega}_1\left(\alpha, \tilde{l}\right) = \tilde{\omega}_1$, given the current limit $\overline{\omega}_1(\alpha, l)$.[6]

The second part characterizes the welfare effects of PMP for small amounts of effective tightening. For a sketch proof, note that the policies $\tilde{l}$ and $Q_1\left(\cdot, \tilde{l}\right)$ lead to identical equilibrium allocations except for the asset price in the boom state. Using this observation and the definition of the gap value in (28), the welfare difference between the two policies can be written as

$$w_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) - w_1^{pl}\left(\alpha, \tilde{l}\right) = \int_0^\infty e^{-\left(\rho + \lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha_{t,1}, \tilde{l}\right)\right) - W\left(Q^*\right)\right) dt. \tag{40}$$

Here, $\alpha_{t,1}$ denotes optimists' wealth share when the economy starts with $\alpha_{0,1} = \alpha$, follows

---

[6] One difference from Proposition 3 is that the policy's effect on the interest rate is more complicated because the price drift $\dot{Q}_{t,1}$ is not necessarily zero. This non-zero drift affects the equilibrium return to capital [cf. Eq. (18)] and thus the equilibrium interest rate. As long as $\tilde{l}$ is in a neighborhood of l, this effect is small and the interest rate in the boom state remains strictly positive (in particular, the policy doesn't violate the zero lower bound). In fact, in the numerical simulations (described below), PMP increases the interest rate.
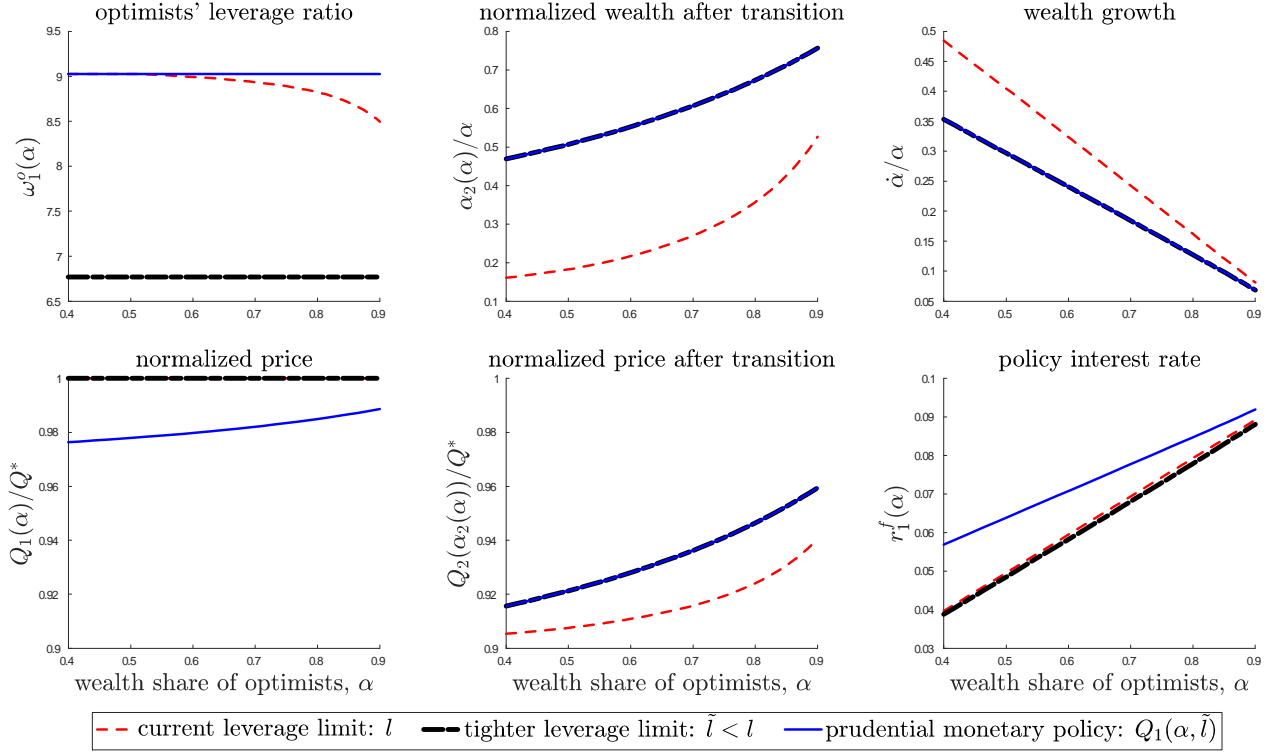
Figure 3: Equilibrium functions in the boom state $s = 1$ for different specifications of the leverage limit and PMP.

policy $\tilde{l}$, and reaches time t without transitioning into recession. Since $W(Q_{t,1}) < W(Q^*)$ for $Q_{t,1} < Q^*$, this expression implies that PMP always yields lower welfare than the equivalent tightening of the leverage limit. However, since $W(Q_{t,1})$ is maximized at $Q_{t,1} = Q^*$, these welfare differences are second order when the prudential policy is used in small doses (so that $Q_{t,1}$ remains close to Q^*). Therefore, as formalized by Eq. (39), the two policies have identical first-order effects on welfare.

5.3. Numerical illustration

We next illustrate the effects of PMP with a numerical example. Suppose optimists' and pessimists' beliefs about the probability of a transition to recession are given by $\lambda_1^o = 0.09 < \lambda_1^p = 0.9$ and the remaining parameters are as described in Appendix A.6. We work with the simple leverage limit function in (38). We assume the current limit barely binds when optimists have half of the wealth share. This amounts to setting: $l = \omega_1^o(0.5, \infty) = 9.03$. The planner would like to tighten this constraint by a quarter, $\tilde{l} = 0.75l = 6.77$, but she cannot control the leverage limit directly. Instead, the planner implements the replicating prudential policy, $Q_1\left(\alpha, \tilde{l}\right)$.

Figure 3 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$. The red dashed lines correspond to the case with the current leverage limit

l but no prudential policy of any kind. The black dash-dotted lines correspond to tightening the leverage limit directly, $\tilde{l} = 0.75l$. Finally, the blue solid lines correspond to implementing this tightening via PMP, $Q_1\left(\alpha, \tilde{l}\right)$.

The top left panel illustrates optimists' leverage ratio as a function of their wealth share for each specification. Optimists have an above-average leverage ratio. The current leverage limit restricts optimists' leverage ratio only slightly (not visible in the figure). The proposed tightening would restrict their leverage ratio considerably more. PMP *raises* optimists' leverage ratio (over the range $\alpha > 0.5$) as it pushes them against the leverage limit.

The top middle panel illustrates optimists' wealth share after transition normalized by their current wealth share, $\alpha_2\left(\alpha\right)/\alpha$. Optimists' wealth share declines after transition, $\alpha_2\left(\alpha\right)/\alpha < 1$. PMP replicates the effect of tightening the leverage limit and therefore increases optimists' wealth share after transition. The top right panel illustrates that this effective tightening slows down the growth of optimists' wealth share if there is no transition.

The bottom left panel illustrates the equilibrium asset price in the boom state normalized by the efficient level. The leverage limit (its current level or hypothetical tightening) leaves the asset price equal to its efficient level. In contrast, PMP reduces the asset price by around 2%. This relatively small decline is able to replicate the effects of a large reduction in optimists' leverage ratio because optimists' initial leverage ratio is high. With high and constrained leverage, small changes in asset prices have large effects on optimists' balance sheets [cf. (36)].

The bottom middle panel illustrates the price after a transition to recession normalized by the efficient level. PMP increases the asset price during the recession. We can gain intuition for this result by comparing this panel with the bottom left panel. By lowering the asset price during the boom, PMP reduces the *asset price decline* after a transition to recession. This smaller decline supports optimists' balance sheets and thus improves the *asset price level* during the recession by around 2%.

The bottom right panel illustrates the equilibrium interest rate. The leverage limit reduces the policy interest rate because it reduces optimists' effective asset demand. In contrast, PMP increases the policy interest rate (by less than 2 percentage points). This reduces the asset price, as illustrated by the bottom left panel, which results in a smaller asset price decline when there is a transition to recession. Equivalently, by raising the interest rate, monetary policy creates room to mitigate the asset price *decline* that results from negative shocks.

Figure 4 simulates the equilibrium variables over time (for each policy specification) for a particular initial wealth share for optimists, α_0, and a particular realization of uncertainty. We take $\alpha_0 = 0.85$, and we consider a path in which the economy transitions into the recession at $t = 0.2$ and recovers from the recession at $t = 0.6$ (other choices lead to qualitatively similar effects). The plots illustrate that PMP raises the asset price in the recession at the cost of reducing it in the boom. In this example, the increase in the asset price level during the recession is greater than the required decline during the boom, but this is not always the case. Regardless of the relative magnitudes, the policy improves welfare (as we will show) because the
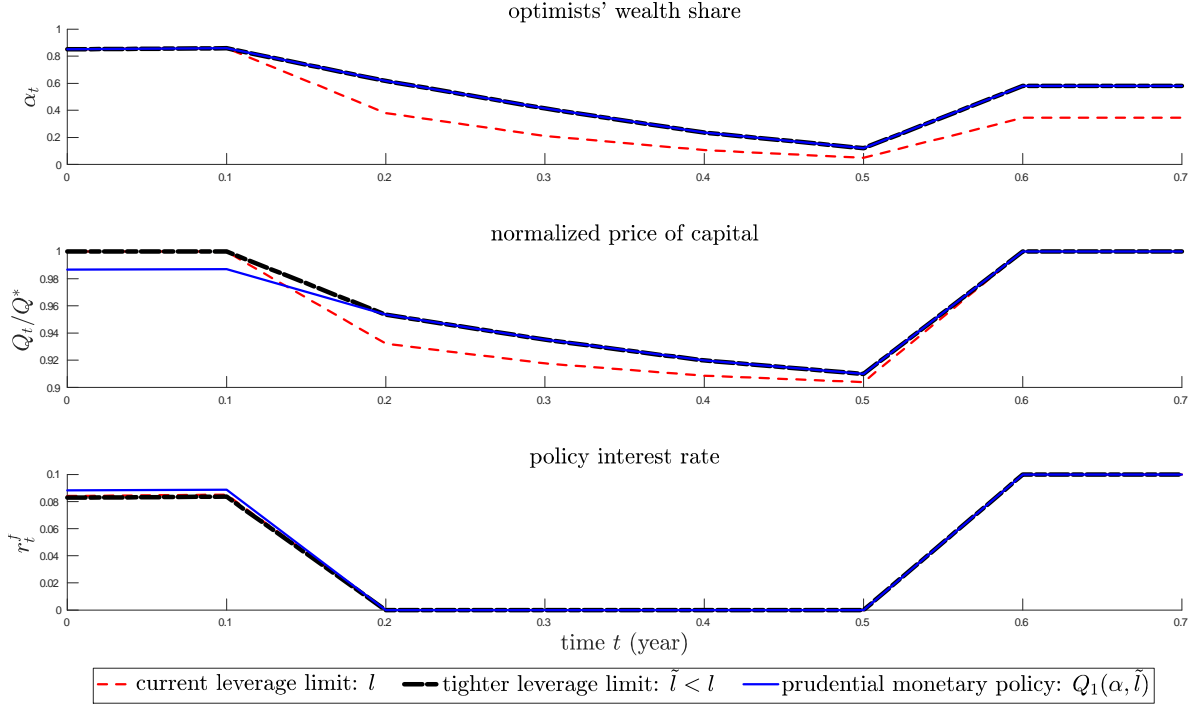
Figure 4: Simulation of the equilibrium path starting with $a_0 = 0.85$ and $s = 1$ for different specifications of the leverage limit and PMP.

asset-price increase in the recession generates first-order benefits, whereas the asset-price decline in the boom generates second-order welfare losses.

Figure 5 illustrates the welfare effects of the policy by plotting the planner's gap value function, $w_1^{pl}(\alpha_0)$ [cf. Eq. (28)]. We take the planner's beliefs to be the average of optimists' and pessimists' beliefs, $\lambda_s^{pl} = (\lambda_s^o + \lambda_s^p)/2$. The black dash-dotted line in Figure 5 illustrates that, if feasible, a direct tightening of the leverage limit would improve the gap value. The solid blue line illustrates that an indirect tightening via PMP also increases the gap value. In fact, for small policy changes, PMP has the same welfare impact as a direct tightening, illustrating the second part of Proposition 4. This can be seen graphically in Figure 5 by comparing the gap values at the point corresponding to the leverage tightening studied above (which we highlight with the vertical dotted line). For small policy changes, welfare losses from the asset price decline during the boom are second order. As the (desired) limit is tightened further, these welfare losses grow larger and PMP becomes less desirable compared to a direct tightening.

6. Optimal prudential monetary policy

So far, we have established that monetary policy can have prudential benefits by effectively tightening an existing leverage limit. In this section, we analyze the determinants of optimal PMP in our setting. We first characterize the optimal prudential policy as the solution to a
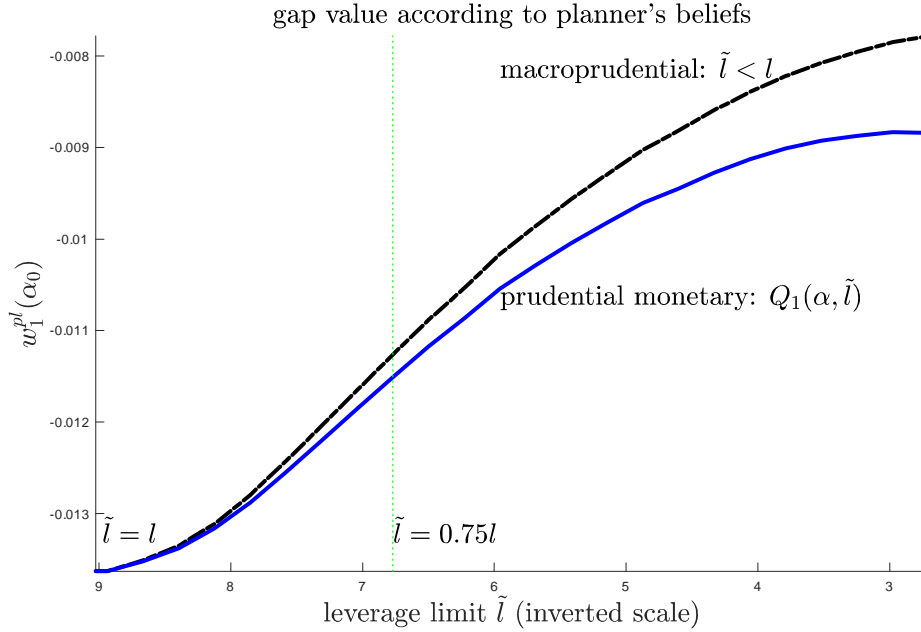
Figure 5: The planner's gap value as a function of the effective leverage ratio starting with $a_0 = 0.85$ and $s = 1$ for a direct tightening (dashed line) and an equivalent tightening via PMP (solid line).

recursive optimization problem. We then solve the problem numerically and investigate the comparative statics of optimal policy.

For each α, suppose the planner sets an arbitrary price level $Q_1 \leq Q^*$ subject to the restriction that the price level weakly declines after the transition. Given Q_1, optimists' wealth share after transition is determined by the function $\alpha_2(\alpha, \overline{\omega}_1, Q_1) \in [0, 1]$. This is a continuous and piecewise differentiable function that is equal to $\alpha_2(\alpha, \infty)$ if optimists' leverage limit does not bind (that is, if $\omega_1^o(\alpha, \infty, Q_1) < \overline{\omega}_1(\alpha)$) and is equal to the solution to (36) if the limit binds. Using this notation, we can recursively formulate the planner's optimization problem in the boom state $s = 1$ as:

$$
\left(\rho + \lambda_1^{pl}\right) w_1^{pl}(\alpha) = \max_{Q_1} W(Q_1) - W(Q^*) + \frac{dw_1^{pl}(\alpha)}{d\alpha} \dot{\alpha} + \lambda_1^{pl} w_2^{pl}(\alpha_2) \tag{41}
$$

$$
\text{where } \dot{\alpha} = \frac{\alpha(1-\alpha)\lambda_1^p}{1-\alpha_2}\left(1 - \frac{\alpha_2}{\alpha}\right)
$$

$$
\alpha_2 = \alpha_2(\alpha, \overline{\omega}_1, Q_1)
$$

$$
\text{and } Q_1 \in [Q_2(\alpha_2(\alpha, \overline{\omega}_1, Q_1)), Q^*].
$$

Here, the second line uses Eq. (35) to describe the evolution of optimists' wealth share absent a transition, $\dot{\alpha} = \frac{d\alpha_{t,1}}{dt}$, as a function of their induced wealth share after transition, $\alpha_2 = \alpha_{t,2}$ (as

well as their current wealth share, $\alpha = \alpha_{t,1}$).[7]

The analytical solution to problem (41) is complicated in part because there might be a discontinuity in the optimal policy function.[8] However, it is straightforward to solve problem (41) numerically. Moreover, we can glean some intuition by considering the local optimality conditions. Specifically, for an interior solution $Q_1 \in (Q_2, Q^*)$, the optimality condition (for decreasing Q_1 further) can be written as:

$$\frac{dW(Q_1)}{dQ_1} = \frac{d\alpha_2}{d(-Q_1)} \left[\lambda_1^{pl} \frac{dw_2^{pl}(\alpha_2)}{d\alpha_2} + \frac{d\dot{\alpha}}{d\alpha_2} \frac{dw_1^{pl}(\alpha)}{d\alpha} \right] \tag{42}$$

$$\text{where } \frac{d\dot{\alpha}}{d\alpha_2} = -\lambda_1^p \frac{(1-\alpha)^2}{(1-\alpha_2)^2}.$$

The left-hand side of Eq. (42) captures the costs of the policy via its impact on the output gap in period 1. This term is positive since $W'(Q_1) > 0$: decreasing the asset price in the boom exacerbates the output gap. The right-hand side captures the welfare effects of the policy via its impact on optimists' wealth share. We have $\frac{d\alpha_2}{d(-Q_1)} > 0$: lowering the asset price increases optimists' wealth share after transition. We also have $\frac{dw_2^{pl}(\alpha_2)}{d\alpha_2} > 0$: increasing optimists' wealth share after transition internalizes aggregate demand externalities and mitigates output gaps. Hence, the first term inside the brackets is positive and captures the static benefits of PMP. On the other hand, we also have $\frac{d\dot{\alpha}}{d\alpha_2} < 0$: if there is no transition, the policy slows down the accumulation of optimists' wealth share. Moreover, we have $\frac{dw_1^{pl}(\alpha)}{d\alpha} > 0$: the reduction in optimists' wealth share in the boom state widens output gaps in a future recession. Therefore, the second term inside the brackets is negative and captures the dynamic costs of PMP.

6.1. Numerical illustration

Figure 6 illustrates the optimal monetary policy corresponding to the numerical example in Section 5.2. As a benchmark, the red dashed lines illustrate the equilibrium without PMP but with the simple leverage limit $\overline{\omega}_1(\alpha, l) = l = 9.03$. Recall that this leverage limit is chosen so that (absent PMP) it binds for optimists when $\alpha < 0.5$ but not when $\alpha \geq 0.5$. The green dotted line in the left panel illustrates the minimum price decline necessary to make the leverage limit bind for optimists—price reductions smaller than this level have no prudential benefits as they are undone by endogenous risk adjustments by optimists.

The blue solid line in the left panel of Figure 6 illustrates the optimal price that solves problem (41). With this parameterization, the planner does not use monetary policy for prudential

---

[7]In problem (41), we ignore the zero lower bound constraint on the interest rate. In numerical solutions (described subsequently), we check and verify that this constraint doesn't bind at the optimal solution.

[8]This discontinuity emerges from the fact that, if the leverage limit doesn't bind absent policy ($\omega_1^o(\alpha, \infty, Q_1^*) < \overline{\omega}_1(\alpha)$), then prudential monetary policy requires a discontinuous decline in asset prices and output. In particular, there might be a threshold level of optimists' wealth share, $\overline{\alpha}$, where the planner is indifferent between setting $Q_1(\overline{\alpha}) < Q_1$ (and using the policy) and setting $Q_1(\overline{\alpha}) = Q_1^*$ (and not using the policy).
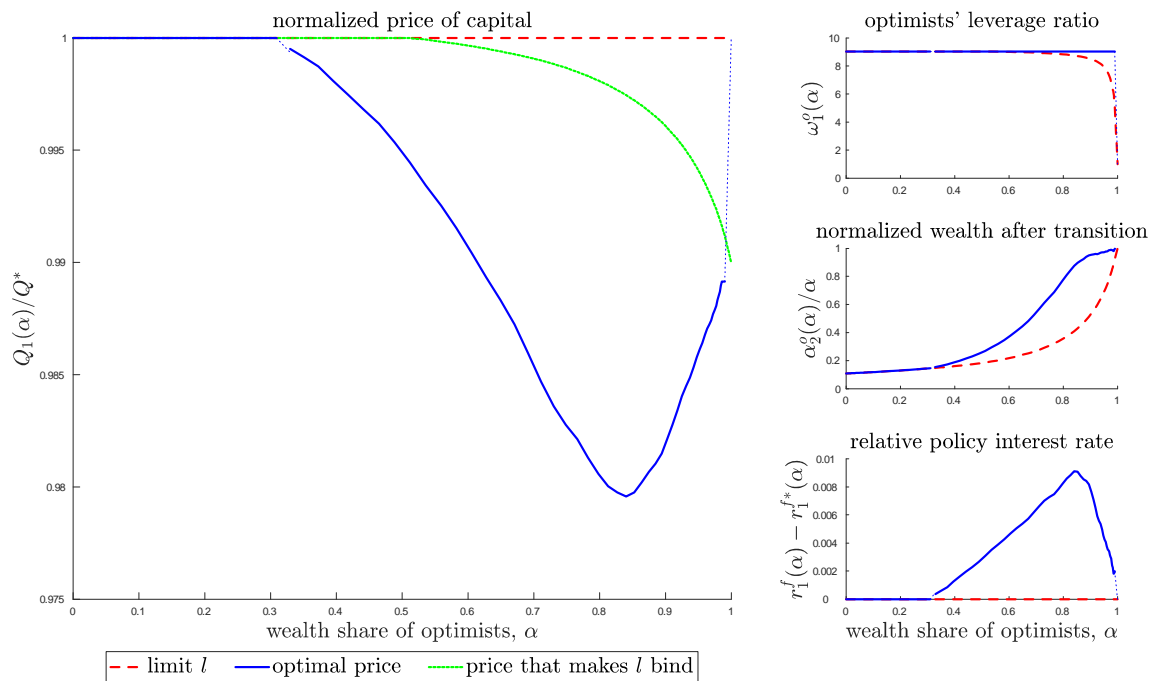
Figure 6: Equilibrium with optimal PMP (blue solid line) and without PMP (red dashed line) given the leverage limit l. The green dotted line in the left panel illustrates the minimum price decline necessary to make optimists' leverage limit bind.

purposes when $\alpha < 0.33$. In this range, the leverage limit is already tight, and tightening it further via PMP does not create large enough benefits to compensate for the costs imposed by slowing down the accumulation of optimists' wealth share [cf. Eq. (42)]. In contrast, the planner uses PMP over the range $\alpha \in [0.33, 0.99]$. Moreover, the degree of tightening relative to the conventional policy benchmark is non-monotonic in the optimists' wealth share. In particular, the planner tightens the policy more as optimists' wealth share increases toward $\alpha = 0.85$ and tightens it less beyond this level. Hence the policy is most useful when optimists' wealth share lies in an intermediate range. Two forces make the policy relatively less attractive for large α. First, since optimal private leverage drops as α rises, the policy becomes costlier as the planner needs to reduce the price even further to make optimists' leverage limit bind and gain some traction (as illustrated by the green dotted line). Second, the policy is less useful because there is less speculation. In fact, for $\alpha \simeq 0.99$, these countervailing forces are strong enough that the planner stops using the policy altogether (as illustrated by the jump in the blue solid line).

Figure 7 illustrates the comparative statics of the optimal policy. To facilitate exposition, we describe the effects for a particular level of optimists' wealth share, $\alpha = 0.85$ (the same wealth share we considered in the previous section). The top panels display the change in the optimal price level as we vary a single parameter. The bottom panels display the change in the optimal interest rate relative to the conventional policy benchmark with output stabilization.
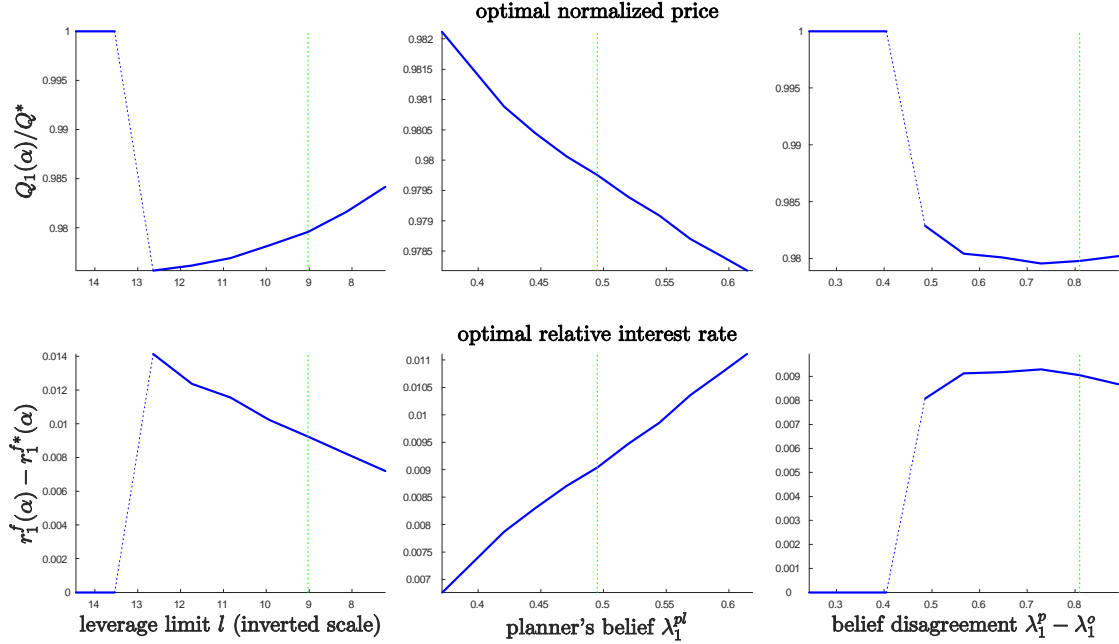
Figure 7: Comparative statics of the optimal PMP price level (top panels) and the interest rate (bottom panels) for $\alpha = 0.85$ with respect to changing the parameter on the x-axis. The vertical dotted lines illustrate the benchmark parameters (used in earlier figures).

The left panels show the effect of changing the leverage limit, l. When the leverage limit is very loose, the planner does not use prudential policy because it is easily undone by optimists, illustrating Proposition 2. There is a threshold leverage limit below which the planner uses monetary policy. Once the leverage limit is below this threshold, tightening it further makes the planner use PMP less. Hence, the leverage limit and PMP are *complements* in the high-l range but they become *substitutes* in the low-l range.

The middle panels illustrate the effect of changing the planner's belief about the probability of transition into recession, λ_1^{pl}. As expected, when the planner believes the recession is more likely, she utilizes PMP more and reduces the asset price by a greater amount.

The right panels show the effect of changing belief disagreements, $\lambda_1^p - \lambda_1^o$ (keeping the mean belief constant at $\frac{\lambda_1^p + \lambda_1^o}{2}$). With greater belief disagreements, the planner is more likely to utilize PMP. Intuitively, disagreements increase speculation (and optimists' risk-exposure), which makes PMP more useful. Conditional on using the policy, the planner does not change the intensity of the policy very much.[9] This insensitivity arises because, once the policy is used, it sets optimists against the leverage limit, which largely decouples equilibrium outcomes from the magnitude of belief disagreements.

[9]In particular, the main effect of greater disagreements is to reduce the threshold level of optimists' wealth share above which the planner uses prudential monetary policy (see Figure 6).

32

7. Prudential policies with "shadow banks"

In practice, a major concern with macroprudential policy is that there are lightly regulated institutions—typically referred to as shadow banks—that can circumvent regulatory constraints. Stein (2013) noted that in these environments PMP might have an advantage over macroprudential policy "because it gets in all of the cracks." We next evaluate the performance of macroprudential policy and PMP in our model when some of the high-valuation agents can avoid regulatory leverage limits. We find that whether PMP is more effective than macroprudential policy depends on the nature of the leverage limits faced by shadow banks.

First consider the case in which shadow banks face a binding leverage limit, e.g., due to financial frictions or self-imposed limits, even though they not constrained by the regulatory limit (see Remark 3). In this case, shadow banks and regular banks are both constrained (although perhaps for different reasons), so our earlier analysis for PMP applies. In particular, PMP can replicate the prudential effects of tightening the leverage limit *for all banks*, whereas macroprudential policy is weaker because a direct regulatory tightening of the leverage limit applies only to regular banks. This broader impact illustrates that PMP can indeed be more effective than macroprudential policy.

Next consider the other extreme case in which shadow banks do not face a binding leverage limit (whereas regular banks face a binding limit). Recall from our earlier analysis that when banks are unconstrained they respond to PMP by increasing their leverage ratio. This suggests that the presence of shadow banks could reduce the effectiveness of PMP as well as macroprudential policy. In particular, it is no longer clear if PMP provides prudential benefits. In the rest of this section, we formally analyze this case and find that PMP remains useful, but it is weakened by the same general equilibrium forces that mitigate macroprudential policy.

Formally, suppose a subset of optimists are not subject to the leverage constraint, $\omega_{1,t} \leq \overline{\omega}_{1,t}$. We refer to these agents as *unconstrained optimists*, and refer to the remaining fraction of optimists as *constrained optimists*. Recall that we view (constrained) optimists as the model counterpart to "banks" (see Remark 2). Therefore, unconstrained optimists are the model counterpart to "shadow banks" that circumvent the regulatory limit and also do not face non-regulatory limits. We let $\beta \in (0,1)$ denote the relative fraction of optimists' wealth that is held by unconstrained optimists. Hence, the wealth share of unconstrained and constrained optimists is given by, respectively, $\alpha\beta$ and $\alpha(1-\beta)$. As before, the total wealth share of optimists (including both types) and pessimists is given by, respectively, α and $1-\alpha$. The rest of the model is unchanged.

To characterize the equilibrium, consider first the recession state $s = 2$. Conditional on the total mass of optimists, α_2, the equilibrium is the same as before. This is because we assume optimists face no constraints from state 2 onwards, which implies that constrained and unconstrained optimists are effectively the same from this point forward. In particular, the equilibrium price in the recession can be written as $Q_{t,2} = Q_2(\alpha_{t,2})$, where $Q_2(\cdot)$ is the price

function characterized earlier [cf. Eq. (26)].

Next consider the equilibrium in the boom state, $s = 1$. In this case, there are two state variables: the total mass of optimists, $\alpha \in (0,1)$, and the fraction of unconstrained optimists, $\beta \in (0,1)$. Therefore, we denote the equilibrium variables as functions of two state variables, in addition to the leverage limit and PMP functions. In particular, $\alpha_2 \left(\alpha, \beta, \overline{\omega}_1 \left(\cdot \right), Q_1 \left(\cdot \right) \right)$ and $\beta_2 \left(\alpha, \beta, \overline{\omega}_1 \left(\cdot \right), Q_1 \left(\cdot \right) \right)$ denote, respectively, the total mass of optimists and the fraction of un-constrained optimists that obtain if there is an instantaneous transition to recession. To simplify the notation, we suppress the dependence of these functions on some or all of their arguments as long as the appropriate arguments are clear from the context.

Much of our earlier analysis applies in this setting. In particular, Eq. (22), which character-izes the growth rate of agents' wealth shares absent a state transition, applies for all agents. In the appendix, we solve the corresponding equations for constrained and unconstrained optimists to obtain the dynamics of α and β as follows:

$$
\begin{aligned}
\frac{\dot{\alpha}}{\alpha} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2} \left(1 - \frac{\alpha_2}{\alpha} \right), \\
\frac{\dot{\beta}}{\beta} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2} \frac{\alpha_2}{\alpha} \left(1 - \frac{\beta_2}{\beta} \right).
\end{aligned}
\tag{43}
$$

Given α_2, optimists' total wealth share follows the same equation as before (cf. Eq. (35)). Given β_2 and α_2, the relative wealth share of unconstrained optimists follows a similar equation. Below, we will verify that the equilibrium features $\alpha_2 < \alpha$ and $\beta_2 < \beta$. Combining this observation with (43) implies $\dot{\alpha} > 0$ and $\dot{\beta} > 0$. Optimists' total wealth share (resp. unconstrained optimists' relative wealth share) grows absent transition to recession, because these agents take on greater risk and earn higher returns compared to pessimists (resp. constrained optimists).

It remains to characterize the functions α_2 and β_2. To this end, note that the portfolio optimality condition (21) holds as equality for unconstrained optimists and as a weak inequality for constrained optimists. Combining these observations, we obtain:

$$
\lambda_1^o \frac{\alpha \left(1 - \beta \right)}{\alpha_2 \left(1 - \beta_2 \right)} \geq \lambda_1^o \frac{\alpha \beta}{\alpha_2 \beta_2} = \lambda_1^p \frac{1 - \alpha}{1 - \alpha_2}.
\tag{44}
$$

Note also that Eq. (20), which relates agents' wealth shares after transition to their leverage ratio, applies for all agents. Using this condition for constrained and unconstrained optimists, we obtain:

$$
\frac{\alpha_2 \left(1 - \beta_2 \right)}{\alpha \left(1 - \beta \right)} = 1 - (1 - \omega_1^{o,reg}) \left(\frac{Q_1}{Q_2 \left(\alpha_2 \right)} - 1 \right)
\tag{45}
$$

$$
\frac{\alpha_2 \beta_2}{\alpha \beta} = 1 - (1 - \omega_1^{o,unreg}) \left(\frac{Q_1}{Q_2 \left(\alpha_2 \right)} - 1 \right).
\tag{46}
$$

Given current α, the current price level Q_1 and the price function after transition $Q_2 \left(\cdot \right)$, the

equilibrium levels of α_2, β_2 (as well as those for $\omega_1^{o,reg}, \omega_1^{o,unreg}$) can be characterized by solving Eqs. $(44-46)$.

Consider the case in which constrained optimists' leverage constraint binds (the other case is the same as in previous sections). In this case, we have $\omega_1^{o,reg} = \overline{\omega}_1$. Substituting this into Eq. (45), we obtain:

$$\frac{\alpha_2 (1 - \beta_2)}{\alpha (1 - \beta)} = 1 - (\overline{\omega}_1 - 1) \left[\frac{Q_1}{Q_2 (\alpha_2)} - 1 \right]. \tag{47}$$

As before, this expression describes constrained optimists' relative wealth share as a function of the leverage limit and the price drop after transition. Solving for β_2 from Eq. (44), and substituting into Eq. (47), we further obtain:

$$\frac{1}{1 - \beta} \left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p} \beta \frac{1 - \alpha_2}{1 - \alpha} \right) = 1 - (\overline{\omega}_1 - 1) \left[\frac{Q_1}{Q_2 (\alpha_2)} - 1 \right]. \tag{48}$$

This equation generalizes Eq. (34) (which we analyzed extensively in previous sections) to cases with $\beta > 0$. In particular, the equation characterizes α_2 given $Q_1, Q_2 (\alpha_2)$ and $\overline{\omega}_1$.

Note also that the left-hand side of Eq. (48) is an increasing function of α_2. Hence, as before, the equation can be visualized as the intersection of two increasing relations between α_2 and Q_2. Under appropriate regularity conditions (relegated to the appendix), there is a unique intersection. The following result considers the benchmark case without PMP, $Q_1 = Q^*$, and establishes the comparative statics of the equilibrium with respect to the fraction of unconstrained optimists, β. The result also establishes the comparative statics with respect to the leverage limit $\overline{\omega}_1$ and generalizes our earlier result about macroprudential policy (Proposition 1) to this setting.

Proposition 5. *Suppose Assumptions 1-2 and A1-A3 hold and that a fraction, $\beta \in (0, 1)$, of optimists' wealth is held by unconstrained optimists that face no leverage limits. Consider the benchmark equilibrium without PMP, $Q_1 (\alpha) = Q^*$. Fix levels $\alpha, \beta \in (0, 1)$ that are associated with some binding leverage limit, $\overline{\omega}_1 (\alpha, \beta) < \omega_1^{o,reg} (\alpha, \beta, \infty)$. Absent transition to recession, α and β follow the dynamics in (43). After transition, α_2 is characterized as the solution to Eq. (48) and β_2 is characterized as the solution to (44). In equilibrium, $\alpha_2 < \alpha, \beta_2 < \beta$ and $\dot{\alpha} > 0, \dot{\beta} > 0$: optimists' total wealth share and unconstrained optimists' relative wealth share shrink after transition to recession and grow absent transition. Moreover, α_2 satisfies the following comparative statics:*

(i) Increasing the relative wealth share of unconstrained optimists, β, decreases optimists' wealth share after transition, $\frac{d\alpha_2(\alpha, \beta, \overline{\omega}_1(\cdot))}{d\beta} < 0$. In the limit as $\beta \to 1$, optimists' wealth share approaches its level in the equilibrium without leverage limits, $\alpha_2 (\alpha, \infty)$.

(ii) Macroprudential policy that decreases the leverage limit increases optimists' wealth share after a transition to recession, $\frac{d\alpha_2(\alpha, \beta, \overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha, \beta)} < 0$. Increasing the relative wealth share of uncon-strained optimists, β, reduces the effectiveness of macroprudential policy, $\frac{\partial}{\partial \beta} \frac{d\alpha_2(\alpha, \beta, \overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha, \beta)} > 0$.

This result verifies the conventional wisdom that the presence of less constrained agents reduces the strength of macroprudential policy. The first part shows that, as the relative wealth share of unconstrained optimists grows, optimists take on greater risk and their wealth share declines by a greater magnitude after transition to recession. The second part shows that (as long as some optimists are constrained, $\beta < 1$) macroprudential policy that tightens leverage limits mitigates the decline in optimists' wealth share but less so than in the earlier setting without unconstrained optimists.

Next consider PMP that lowers the current asset price level, $Q_1(\alpha, \beta) \leq Q^*$. As illustrated by Eq. (47), PMP reduces constrained optimists' exposure to recession. As illustrated by Eq. (48), this in turn increases the wealth share of optimists after transition to recession, α_2. Eq. (48) further suggests that, as before, PMP affects the equilibrium in much the same way as a decrease in $\overline{\omega}_1$. The following result verifies this intuition and generalizes our main result showing that monetary policy can replicate the prudential effects of directly tightening a leverage limit (cf. Proposition 3).

Proposition 6. *Suppose Assumptions 1-2 and A1-A3 hold and that a fraction, $\beta \in (0,1)$, of optimists' wealth is held by unconstrained optimists that face no leverage limits. Fix some $\alpha, \beta \in (0,1)$ and consider the setup of Proposition 3. In particular, consider an alternative leverage limit $\tilde{\omega}_1(\cdot)$ that agrees with $\overline{\omega}_1(\cdot)$ everywhere except for (α, β) and that satisfies $\tilde{\omega}_1(\alpha, \beta) < \min(\overline{\omega}_1(\alpha, \beta), \omega_1^{o,reg}(\alpha, \beta\infty))$. Then:*

(i) There exists $\tilde{Q}_1(\alpha, \beta) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on constrained and unconstrained optimists' wealth shares after transition as the alternative leverage limit (without PMP):

$$\alpha_2\left(\alpha, \beta, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2(\alpha, \beta, \tilde{\omega}_1) \ \ and \ \beta_2\left(\alpha, \beta, \overline{\omega}_1, \tilde{Q}_1\right) = \beta_2(\alpha, \beta, \tilde{\omega}_1).$$

Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial \tilde{Q}_1(\alpha, \beta)}{\partial \tilde{\omega}_1(\alpha, \beta)} > 0$.

(ii) PMP requires setting a higher interest rate than the benchmark without policy:

$$r_1^f\left(\alpha, \beta, \overline{\omega}_1, \tilde{Q}_1\right) > r_1^f(\alpha, \beta, \overline{\omega}_1).$$

Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f\left(\alpha, \beta, \overline{\omega}_1, \tilde{Q}_1\right)}{\partial \tilde{\omega}_1(\alpha, \beta)} < 0$.

The sketch-proof of this result is the same as in Proposition 3. In particular, the monetary authority can choose $\tilde{Q}_1$ so that optimists' total wealth share and the equilibrium price in the recession settle at the same level as if the regulator had directly tightened the leverage limit. In fact, conditional on optimists' wealth share α_2, the replicating $\tilde{Q}_1$ that the planner needs to set is characterized as the solution to the same equation (37) as in our earlier analysis.
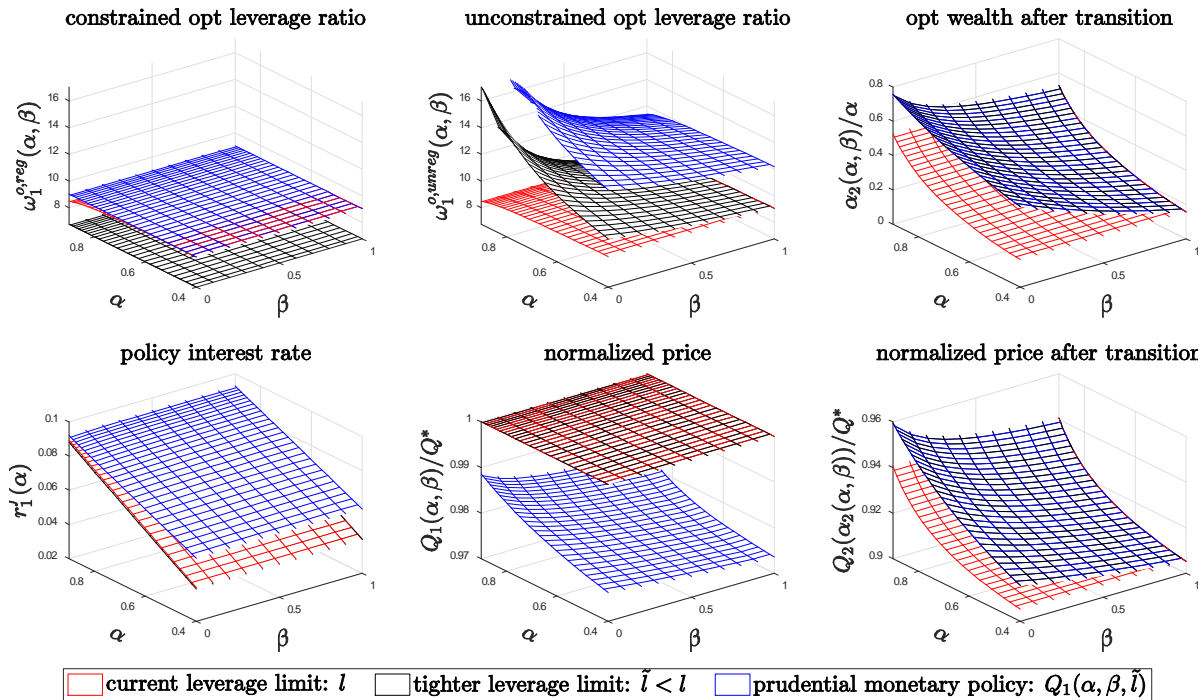
constrained opt leverage ratio   **unconstrained opt leverage ratio**   **opt wealth after transition**

policy interest rate   **normalized price**   **normalized price after transition**

current leverage limit: l tighter leverage limit: $\tilde{l} < l$ prudential monetary policy: $Q_1(\alpha, \beta, \tilde{l})$

Figure 8: Equilibrium functions in the boom state $s = 1$ with unconstrained optimists for different specifications of the leverage limit and PMP. β is the fraction of optimists' wealth held by unconstrained optimists.

7.1. Numerical illustration

We next illustrate the effects of macroprudential policy and PMP in the presence of unconstrained optimists. Consider the same example we analyzed in Section 5.3. In particular, the current leverage limit barely binds when optimists have half of the wealth share. The planner would like to tighten the existing limit by a quarter, $\tilde{l} = 0.75l$. However, she cannot control the leverage limit directly. Instead, the planner implements the replicating prudential policy, $Q_1\left(\alpha, \beta, \tilde{l}\right)$.

Figure 8 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$ and $\beta \in [0, 1]$. The lines corresponding to $\beta = 0$ match the earlier equilibria without unconstrained optimists (also plotted in Figure 3). The rest of the surfaces illustrate the effect of unconstrained optimists.

First consider the effect of macroprudential policy that tightens leverage limits: specifically, compare the benchmark with the current limit (illustrated with red lines) with a direct tightening of the limit (illustrated with black lines). The top two left panels show constrained and unconstrained optimists' leverage ratios, respectively. In the benchmark, constrained and unconstrained optimists have similar leverage ratios (since the leverage limit barely binds). The proposed tightening of the leverage limit reduces constrained optimists' leverage ratio while

raising unconstrained optimists' leverage ratio. Intuitively, tightening the leverage limit reduces financial stability risk, since it increases asset prices after transition to recession. Unconstrained optimists respond by taking greater risks.

The top right panel illustrates optimists' wealth share after transition to recession. Macro-prudential policy improves optimists' wealth share in the recession but less so than in the case without unconstrained optimists ($\beta = 0$), illustrating Proposition 5. Intuitively, since unconstrained optimists respond to the policy by increasing their risks, they reduce (but do not fully eliminate) the effectiveness of macroprudential policy. Consequently, macroprudential policy improves asset prices in the recession but less so than in the case without unconstrained optimists.

Next consider the PMP (illustrated with blue lines) that replicates the prudential effects of a direct tightening of the leverage limit. The two panels in the bottom left show that PMP replicates the direct tightening by increasing the interest rate and lowering asset prices during the boom, illustrating Proposition 6. The two panels in the top left show that PMP increases the leverage ratio of constrained optimists (as it pushes them against the leverage limit) *and* the leverage ratio of unconstrained optimists. In fact, unconstrained optimists respond by increasing their leverage ratio even more than when the planner directly tightens the leverage limit. These agents obtain the same wealth share after transition, $\alpha_2\beta_2$, as in direct tightening (see Proposition 6). However, they now achieve this outcome by taking on greater leverage since the price drop after transition is smaller (see Eq. (46)).

These results illustrate that, when some high-valuation agents are not subject to any (regulatory or non-regulatory) leverage limit, PMP can still replicate the financial stability benefits of macroprudential policy. However, in this setting PMP is subject to similar limitations as macroprudential policy: unconstrained agents respond to the policy by increasing their leverage and risk taking. This finding is consistent with recent empirical evidence showing that a contractionary monetary policy shock *increases* lending by shadow banks (see Elliott et al. (2019); Drechsler et al. (2019)).

8. Final Remarks

We propose a model of asset price booms with speculation that may justify using PMP to reduce the severity of future recessions. PMP aims to reduce the social cost of concentrating risk in leveraged, high-valuation agents ("optimists" or "banks"). The policy achieves this goal by lowering the asset price level during the boom, which reduces the asset price decline after a transition to recession. This reduction supports highly-levered agents' balance sheets in the recession, which in turn raises asset prices (and hence further reduces the price drop) and softens the recession.

An equivalent interpretation is that PMP raises the interest rate to increase the available "ammunition" for the next recession. This concept has little meaning in most macro models

where all that matters during a recession is the level of interest rates. By contrast, our framework emphasizes the importance of the size of interest rate cuts as the economy transitions from boom to recession. A larger interest rate cut is useful because it mitigates the asset price *decline* as the economy transitions to recession. A smaller asset price decline is preferable because it improves highly-levered agents' wealth share, which is a key state variable that determines the severity of the recession.

Our main insight can be applied beyond the specific binary-state context of our model. For example, in practice, large recessions are often preceded by minor slowdowns, at which time central banks need to decide how quickly to cut interest rates. Our analysis suggests that, if the slowdown is associated with significant financial speculation, then it may be worth delaying interest rate cuts. By doing so, the central bank effectively keeps its ammunition for a larger recession in which monetary policy becomes constrained.

References

Acharya, V., Naqvi, H., 2018. On reaching for yield and the coexistence of bubbles and negative bubbles. Journal of Financial Intermediation.

Adrian, T., de Fontnouvelle, P., Yang, E., Zlate, A., 2017. Macroprudential policy: a case study from a tabletop exercise. Economic Policy Review 23 (1), 1âĂŞ30.

Adrian, T., Liang, N., 2018. Monetary policy, financial conditions, and financial stability. International Journal of Central Banking.

Bernanke, B. S., Kuttner, K. N., 2005. What explains the stock market's reaction to federal reserve policy? The Journal of finance 60 (3), 1221–1257.

Borio, C., Zhu, H., 2012. Capital regulation, risk-taking and monetary policy: a missing link in the transmission mechanism? Journal of Financial stability 8 (4), 236–251.

Borio, C. E., 2014. Monetary policy and financial stability: what role in prevention and recovery?

Borio, C. E., Drehmann, M., 2009. Assessing the risk of banking crises–revisited. BIS Quarterly Review, March.

Brunnermeier, M. K., Simsek, A., Xiong, W., 2014. A welfare criterion for models with distorted beliefs. The Quarterly Journal of Economics 129 (4), 1753–1797.

Caballero, R. J., Simsek, A., 2017. A risk-centric model of demand recessions and macroprudential policy. NBER working paper no. 23614.

Campbell, J. Y., 2014. Empirical asset pricing: Eugene fama, lars peter hansen, and robert shiller. The Scandinavian Journal of Economics 116 (3), 593–634.

Cao, D., 2017. Speculation and financial wealth distribution under belief heterogeneity. The Economic Journal.

Cochrane, J. H., 2011. Presidential address: Discount rates. The Journal of finance 66 (4), 1047–1108.

Collard, F., Dellas, H., Diba, B., Loisel, O., 2017. Optimal monetary and prudential policies. American Economic Journal: Macroeconomics 9 (1), 40–87.

Curdia, V., Woodford, M., 2010. Credit spreads and monetary policy. Journal of Money, credit and Banking 42, 3–35.

Dávila, E., Korinek, A., 2017. Pecuniary externalities in economies with financial frictions. The Review of Economic Studies 85 (1), 352–395.

Drechsler, I., Savov, A., Schnabl, P., 2019. How monetary policy shaped the housing boom. NBER working paper no. 25964.

Elliott, D., Meisenzahl, R. R., Peydró, J.-L., 2019. Nonbanks, banks, and monetary policy: Us loan-level evidence since the 1990s.

Farhi, E., Werning, I., 2016. A theory of macroprudential policies in the presence of nominal rigidities. Econometrica 84 (5), 1645–1704.

Fostel, A., Geanakoplos, J., 2008. Leverage cycles and the anxious economy. The American Economic Review 98 (4), 1211–1244.

Geanakoplos, J., 2010. The leverage cycle. NBER macroeconomics annual 24 (1), 1–66.

Gertler, M., Karadi, P., 2015. Monetary policy surprises, credit costs, and economic activity. American Economic Journal: Macroeconomics 7 (1), 44–76.

Gilchrist, S., López-Salido, D., Zakrajšek, E., 2015. Monetary policy and real borrowing costs at the zero lower bound. American Economic Journal: Macroeconomics 7 (1), 77–109.

Gourio, F., Kashyap, A. K., Sim, J. W., 2018. The trade offs in leaning against the wind. IMF Economic Review 66 (1), 70–115.

Hanson, S. G., Stein, J. C., 2015. Monetary policy and long-term real rates. Journal of Financial Economics 115 (3), 429–448.

Harrison, J. M., Kreps, D. M., 1978. Speculative investor behavior in a stock market with heterogeneous expectations. The Quarterly Journal of Economics 92 (2), 323–336.

Heimer, R., Simsek, A., 2018. Should retail investorsâĂŹ leverage be limited? Journal of Financial Economics.

Iachan, F. S., Nenov, P. T., Simsek, A., 2015. The choice channel of financial innovation. NBER working paper no. 21686.

IMF, 2015. Monetary policy and financial stability. IMF Staff Report.

Jordà, Ò., Schularick, M., Taylor, A. M., 2013. When credit bites back. Journal of Money, Credit and Banking 45 (s2), 3–28.

Kiyotaki, N., Moore, J., 1997. Credit cycles. Journal of political economy 105 (2), 211–248.

Kocherlakota, N., 2014. Discussion of 2014 usmpf monetary policy report. Tech. rep., Federal Reserve Bank of Minneapolis.

Korinek, A., Simsek, A., 2016. Liquidity trap and excessive leverage. American Economic Review 106 (3), 699–738.

Lian, C., Ma, Y., Wang, C., 2018. Low interest rates and risk taking: Evidence from individual investment decisions. Review of Financial Studies.

Lintner, J., 1969. The aggregation of investor's diverse judgments and preferences in purely competitive security markets. Journal of financial and quantitative analysis 4 (04), 347–400.

Lorenzoni, G., 2008. Inefficient credit booms. The Review of Economic Studies 75 (3), 809–833.

Maddaloni, A., Peydró, J.-L., 2011. Bank risk-taking, securitization, supervision, and low interest rates: Evidence from the euro-area and the us lending standards. the review of financial studies 24 (6), 2121–2165.

Martinez-Miera, D., Repullo, R., 2019. Monetary policy, macroprudential policy, and financial stability.

Miller, E. M., 1977. Risk, uncertainty, and divergence of opinion. The Journal of finance 32 (4), 1151–1168.

Morris, S., Shin, H. S., 2014. Risk-taking channel of monetary policy: A global game approach. Unpublished working paper, Princeton University.

Rajan, R. G., 2006. Has finance made the world riskier? European Financial Management 12 (4), 499–533.

Rognlie, M., Shleifer, A., Simsek, A., 2018. Investment hangover and the great recession. American Economic Journal: Macroeconomics 10 (2), 113–53.

Scheinkman, J. A., Xiong, W., 2003. Overconfidence and speculative bubbles. Journal of political Economy 111 (6), 1183–1220.

Simsek, A., 2013a. Belief disagreements and collateral constraints. Econometrica 81 (1), 1–53.

Simsek, A., 2013b. Speculation and risk sharing with new financial assets. The Quarterly journal of economics 128 (3), 1365–1396.

Smets, F., 2014. Financial stability and monetary policy: How closely interlinked? International Journal of Central Banking 10 (2), 263–300.

Stein, J., 2012. Monetary policy as financial stability regulation. The Quarterly Journal of Economics 127 (1), 57–95.

Stein, J., 2013. Overheating in credit markets: Origins, measurement, and policy responses: speech given to the symposium on restoring household financial stability after the great recession, federal reserve bank of st. louis, st. louis, missouri, february 07, 2013. Tech. rep.

Stein, J., 2014. Incorporating financial stability considerations into a monetary policy framework: a speech at the international research forum on monetary policy, washington, dc, march 21, 2014. Tech. rep.

Svensson, L. E., 2017. Cost-benefit analysis of leaning against the wind. Journal of Monetary Economics 90, 193–213.

Svensson, L. E., 2018. The future of monetary policy and macroprudential policy. In: The Future of Central Banking: An ECB Colloquium Held in Honour of Vitor Constancio.

Weidmann, J., 2018. The relationship between monetary and macroprudential policies - black and white or shades of grey?: dinner speech at the annual meeting of the cebra, frankfurt am main, 20 august 2018. Tech. rep., Central Bank Research Association (CEBRA).

Woodford, M., 2012. Inflation targeting and financial stability. NBER working paper no. 17967.

A. Appendix: Omitted derivations

This appendix presents the derivations and proofs omitted from the main text.

A.1. Omitted derivations in Section 2

A.1.1. Recursive formulation of the portfolio problem

We start by deriving the investors' optimality conditions. Recall that the investor's portfolio problem is given by (7). The HJB equation corresponding to this problem is

$$\rho V_{t,s}^i \left(a_{t,s}^i \right) = \max_{c,\omega} \log c + \frac{\partial V_{t,s}^i}{\partial a} \left(a_{t,s}^i \left(r_{t,s}^f + \omega \left(r_{t,s} - r_{t,s}^f \right) \right) - c \right) \tag{A.1}$$
$$+ \frac{\partial V_{t,s}^i \left(a_{t,s}^i \right)}{\partial t} + \lambda_s^i \left(V_{t,s'}^i \left(a_{t,s}^i \left(1 + \omega \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \right) \right) - V_{t,s}^i \left(a_{t,s}^i \right) \right)$$

$$\text{s.t. } \omega \leq \overline{\omega}_{t,1} \text{ if } s = 1.$$

In view of log utility, the solution has the functional form

$$V_{t,s}^i \left(a_{t,s}^i \right) = \frac{\log \left(a_{t,s}^i / Q_{t,s} \right)}{\rho} + v_{t,s}^i. \tag{A.2}$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investor's consumption proportionally at all times and in all states. The second term, $v_{t,s}^i$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^i = Q_{t,s}$). This functional form also implies

$$\frac{\partial V_{t,s}^i}{\partial a} = \frac{1}{\rho a_{t,s}^i}.$$

The first order condition for c then implies Eq. (14) in the main text. The first order condition for ω implies

$$\frac{\partial V_{t,s}^i}{\partial a} a_{t,s}^i \left(r_{t,s} - r_{t,s}^f \right) + \lambda_s^i \frac{\partial V_{t,s'}^i \left(a_{t,s'}^i \right)}{\partial a} a_{t,s}^i \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0,$$

with inequality only if $s = 1$ and $\omega = \overline{\omega}_{t,1}$. After substituting for $\frac{\partial V_{t,s}^i}{\partial a}$ and $\frac{\partial V_{t,s'}^i}{\partial a}$ and rearranging terms, this relation implies

$$r_{t,s} - r_{t,s}^f + \lambda_s^i \frac{a_{t,s}^i}{a_{t,s'}^i} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0,$$

with inequality only if $s = 1$ and $\omega = \overline{\omega}_{t,1}$. After substituting $a_{t,s}^i = \alpha_{t,s}^i Q_{t,s} k_{t,s}$ [cf. Eq. (19)], this gives Eq. (21) in the main text.

A.1.2. Evolution of investors' wealth share

We next derive the evolution of investors' wealth shares. After substituting optimal consumption from (14) into the budget constraint in problem (7), type i investors' wealth evolves according to

$$\frac{da_{t,s}^i/dt}{a_{t,s}^i} = r_{t,s}^f + \omega_{t,s}^i \left(r_{t,s} - r_{t,s}^f \right) - \rho.$$

Combining this with Eq. (9), aggregate wealth evolves according to

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = r_{t,s}^f + \left(r_{t,s} - r_{t,s}^f \right) - \rho.$$

Combining these expressions with $\alpha_{t,s}^i = \frac{a_{t,s}^i}{Q_{t,s}k_{t,s}}$ [cf. Eq. (19)], we obtain:

$$\frac{\dot{\alpha}_{t,s}^i}{\alpha_{t,s}^i} = \left(\omega_{t,s}^i - 1 \right) \left(r_{t,s} - r_{t,s}^f \right). \tag{A.3}$$

Next recall that the portfolio optimality condition (21) holds with equality for pessimists. Applying this equation, we obtain:

$$r_{t,s} - r_{t,s}^f = -\lambda_s^p \frac{\alpha_{t,s}^p}{\alpha_{t,s'}^p} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{A.4}$$

Likewise, applying Eq. (20) for type i investors, we obtain:

$$\omega_{t,s}^i - 1 = \left(\frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i} - 1 \right) \frac{Q_{t,s'}}{Q_{t,s'} - Q_{t,s}}. \tag{A.5}$$

Substituting Eqs. $(A.4)$ and $(A.5)$ into Eq. $(A.3)$, we obtain Eq. (22) in the main text.

A.2. Omitted derivations in Section 3

A.2.1. Equilibrium in the recession and the recovery states

As we describe in the main text, for the rest of the analysis we often simplify the notation by dropping the subscript o from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha_{t,s}^o.$$

Pessimists' wealth share is the complement of this expression, $\alpha_{t,s}^p = 1 - \alpha_{t,s}$.

We next present the details of our characterization of equilibrium for the recession and recovery states, $s \in \{2, 3\}$. We assume the following:

Assumption A1. $\delta(0) - \left(\rho + \lambda_2^i \right) < g_2 < \delta(\eta^*) - \rho < g_3.$

With this assumption, we conjecture an equilibrium in which the recovery state $s = 3$ features positive interest rates, efficient asset prices, and efficient factor utilization, $r_{t,3}^f > 0, Q_{t,3} = Q^*$ and $\eta_{t,3} = \eta^*$. The recession state $s = 2$ features an interest rate of zero, lower asset prices, and inefficient factor utilization, $r_{t,2}^f = 0, Q_{t,2} < Q^*$ and $\eta_{t,2} < \eta^*$. We will show that the equilibrium price in the recession state can be

represented as a strictly increasing function of optimists' wealth share: $Q_{t,2} = Q_2 (\alpha_{t,2})$ where $Q_2 (\cdot)$ is a strictly increasing function.

Note that for $s \in \{2,3\}$ the leverage limit doesn't bind. Therefore, Eq. (23) applies. Combining Eqs. (21) and (23), we obtain:

$$r_{t,s} - r_{t,s}^f + \overline{\lambda}_{t,s} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} = 0. \tag{A.6}$$

In particular, the risk premium is determined by the weighted-average belief, $\overline{\lambda}_{t,s}$.

Equilibrium in the recovery state $s = 3$. In the recovery state, there is no speculation since $\lambda_3^i = 0$ for each i. Substituting this transition probability into Eq. (A.6), we find that the risk premium is zero, $r_{t,3} - r_{t,3}^f = 0$. After substituting for the market return from Eq. (18), and using $\dot{Q}_{t,3} = 0$ (since $Q_{t,3} = Q^*$ is constant), we obtain:

$$r_{t,3}^f = \rho + g_3 - \delta (\eta^*) > 0. \tag{A.7}$$

The inequality follows from Assumption A1. Hence, in the recovery state, the interest rate is constant and strictly positive and the equilibrium asset price and factor utilization levels are efficient.

Equilibrium in the recession state $s = 2$. In this state, there is some speculation since investors have heterogeneous beliefs, $\lambda_2^o > \lambda_2^p$ [cf. Assumption 1]. Substituting Eq. (18) into Eq. (A.6) and using the conjecture $Q_{t,3} = Q^*$, we obtain Eq. (25) in the main text. Substituting the conjecture $r_{t,2}^f = 0$, we further obtain:

$$\rho + g_2 - \delta \left(\frac{Q_{t,2}}{Q^*} \eta^* \right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2} \left(1 - \frac{Q_{t,2}}{Q^*} \right) = 0. \tag{A.8}$$

Next consider the extreme cases $\alpha_{t,2} \in \{0,1\}$. These cases are the same as if there is a single belief type $i \in \{o,p\}$. In particular, since there is no speculation, the price is constant within the state, that is: $Q_{t,2} \equiv Q_2^i$ and thus $\dot{Q}_{t,2} = 0$. Therefore, Eq. (A.8) can be written as

$$\rho + g_2 - \delta \left(\frac{Q_2^i}{Q^*} \eta^* \right) + \lambda_2^i \left(1 - \frac{Q_2^i}{Q^*} \right) = 0.$$

Under Assumption A1, there exists a solution that satisfies $Q_2^i \in (0, Q^*)$. This describes the equilibrium price in the recession state if all investors share type i investors' beliefs. Using $\lambda_2^o > \lambda_2^p$ (Assumption 1), it is easy to check that $Q_2^o > Q_2^p$. In particular, the price is greater under optimists' beliefs than under pessimists' beliefs.

Next consider the intermediate cases, $\alpha_{t,2} \in (0,1)$. In this case we combine Eq. (A.8) with Eq. (24) for state $s = 2$ to obtain a system of differential equations for $(\alpha_{t,2}, Q_{t,2})$:

$$\rho + g_2 - \delta \left(\frac{Q_{t,2}}{Q^*} \eta^* \right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2} \left(1 - \frac{Q_{t,2}}{Q^*} \right) = 0, \tag{A.9}$$

$$\dot{\alpha}_{t,2} = -\alpha_{t,2} (1 - \alpha_{t,2}) \Delta\lambda_2^o.$$

This is similar to the differential equation system for the recession state in Caballero and Simsek (2017). Following similar steps, we show that the system is saddle path stable: for any $\alpha_{t,2}$, there exists a unique equilibrium price level $Q_{t,2} \in [Q_2^p, Q_2^o)$ such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} Q_{t,2} = Q_2^p$. Since the system is stationary, the solution can be written as a function of optimists'

normalized capital price in high-risk-premium state

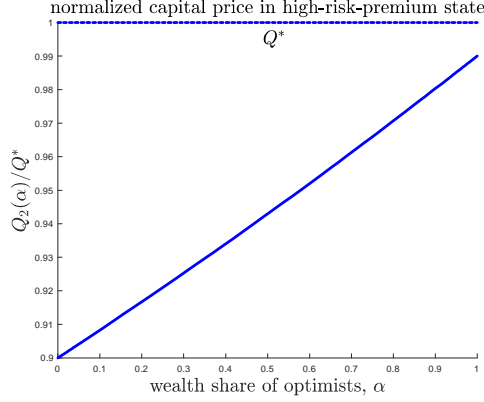$Q_2(\alpha)/Q^*$

wealth share of optimists, α

Figure 9: Equilibrium price of capital in the recession state $s = 2$. The dotted line illustrates the frictionless price level, Q^*.

wealth share, $Q_{t,2} = Q_2(\alpha)$. In Caballero and Simsek (2017), we show that $Q_2(\alpha)$ is strictly increasing in α. Since $Q_2^p < Q_2^o < Q^*$, this establishes Eq. (26) in the main text.

For a numerical solution, we convert the differential equation in $(A.9)$ into a differential equation in α-domain. In particular, differentiating $Q_{t,2} = Q_2(\alpha_{t,2})$ with respect to time, we obtain:

$$\dot{Q}_{t,2} = Q_2'(\alpha_{t,2})\,\dot{\alpha}_{t,2}.$$

Combining this with Eq. $(A.9)$, we obtain:

$$\frac{Q_2'(\alpha)}{Q_2(\alpha)} = \frac{1}{\alpha(1-\alpha)\Delta\lambda_2^o}\left(\rho + g_2 - \delta\left(\frac{Q_2(\alpha)}{Q^*}\eta^*\right) + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2(\alpha)}{Q^*}\right)\right).$$

The equilibrium price function is the solution to this system subject to the boundary conditions $Q_2(0) = Q_2^p$ and $Q_2(1) = Q_2^o$. Figure 9 illustrates the solution for a particular parameterization.

A.2.2. Value functions in equilibrium

We next characterize investors' equilibrium expected values and derive the gap value that we use in the main text. Let the superscript $b \in \{o, p, pl\}$ denote the belief corresponding to optimists, pessimists, or the planner. Let $i \in \{o, p\}$ denote type i investors. We let $V_{t,s}^{i,b}\left(a_{t,s}^i\right)$ denote type i investors' expected value when she has wealth $a_{t,s}^i$, evaluated according to type b belief. In view of log utility, we conjecture the following version of Eq. $(A.2)$:

$$V_{t,s}^{i,b}\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i/Q_{t,s}\right)}{\rho} + v_{t,s}^{i,b}. \tag{A.10}$$

Note that this function implies $\frac{\partial V_{t,s}^{i,b}}{\partial a} = \frac{1}{\rho a_{t,s}^i}$. Using this expression as well as $c_{t,s}^i = \rho a_{t,s}^i$, we obtain the following version of the HJB equation $(A.1)$:

$$\rho V_{t,s}^{i,b}\left(a_{t,s}^i\right) - \frac{\partial V_{t,s}^i\left(a_{t,s}^{i,b}\right)}{\partial t} = \log \rho a_{t,s}^i + \frac{1}{\rho}\left(r_{t,s}^f + \omega_{t,s}^i\left(r_{t,s} - r_{t,s}^f\right) - \rho\right) \tag{A.11}$$
$$+ \lambda_s^b\left(V_{t,s'}^i\left(a_{t,s}^i\left(1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)\right) - V_{t,s}^i\left(a_{t,s}^i\right)\right).$$

Note that we evaluate the value function along the equilibrium path and according to transition probability λ_s^b.

Substituting Eq. $(A.10)$ into Eq. $(A.11)$, we obtain a differential equation for the normalized value:

$$\rho v_{t,s}^{i,b} - \frac{\partial v_{t,s}^{i,b}}{\partial t} = \log \rho + \log Q_{t,s} + \frac{1}{\rho}\left(r_{t,s} - \rho - \frac{\dot{Q}_{t,s}}{Q_{t,s}} + \left(\omega_{t,s}^i - 1\right)\left(r_{t,s} - r_{t,s}^f\right) + \lambda_s^b \log\left(1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}\right)\right)$$
$$+ \lambda_s^b\left(v_{t,s'}^{i,b} - v_{t,s}^{i,b}\right).$$

To simplify this expression, we substitute $r_{t,s} = \rho + \frac{\dot{Q}_{t,s}}{Q_{t,s}} + g_s - \delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right)$ using Eq. (18). We also substitute for $\left(\omega_{t,s}^i - 1\right)\left(r_{t,s} - r_{t,s}^f\right) = \frac{\dot{\alpha}_{t,s}^i}{\alpha_{t,s}^i}$ from Eq. $(A.3)$. Finally, we substitute for $1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} = \frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i}$ using Eq. (20). After these substitutions, we obtain:

$$\rho v_{t,s}^{i,b} - \frac{\partial v_{t,s}^{i,b}}{\partial t} = \log \rho + \log Q_{t,s} + \frac{1}{\rho}\left(g_s - \delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right) + \frac{\dot{\alpha}_{t,s}^i}{\alpha_{t,s}^i} + \lambda_s^b \log\left(\frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i}\right)\right) \tag{A.12}$$
$$+ \lambda_s^b\left(v_{t,s'}^{i,b} - v_{t,s}^{i,b}\right).$$

We have thus characterized the normalized value function, $v_{t,s}^{i,b}$, as a solution to the differential equation in $(A.12)$. This equation applies for any beliefs $b \in \{o, p, pl\}$, including investors' own beliefs $b = i$, and it applies regardless of whether the leverage limit binds. The terms that feature $Q_{t,s}$ capture potential welfare losses due to inefficient factor utilization. The term g_s captures the welfare effect of expected growth. The term $\frac{\dot{\alpha}_{t,s}^i}{\alpha_{t,s}^i} + \lambda_s^b \log\left(\frac{\alpha_{t,s'}^i}{\alpha_{t,s}^i}\right)$ captures the welfare effect of speculation that reshuffles investors' wealth shares across states.

As we describe in the main text, we decompose the normalized value into two components [cf. (27)]:

$$v_{t,s}^{i,b} = v_{t,s}^{i*,b} + w_{t,s}^{i,b},$$

Here, $v_{t,s}^{i*,b}$ is *the frictionless value function*, which is found by solving Eq. $(A.12)$ with $Q_{t,s} = Q^*$ for each t, s. This captures all determinants of welfare except for suboptimal factor utilization (including the benefits/costs from speculation). The residual, $w_{t,s}^b$, corresponds to *the gap value function*. This captures the welfare losses due to suboptimal factor utilization evaluated according to investors' preferences (and type b beliefs).

To further characterize the gap value, note that $v_{t,s}^{i,b}$ and $v_{t,s}^{i*,b}$ both solve Eq. $(A.12)$ with $Q_{t,s}$ and $Q_{t,s} = Q^*$, respectively. Taking the difference of these equations, and using $w_{t,s}^{i,b} = v_{t,s}^{i,b} - v_{t,s}^{i*,b}$, we obtain

Eq. (28) in the main text, which we replicate for ease of exposition:

$$\rho w_{t,s}^b - \frac{\partial w_{t,s}^b}{\partial t} = W(Q_{t,s}) + \lambda_s^b \left(w_{t,s'}^b - w_{t,s}^b \right),$$

$$\text{where } W(Q_{t,s}) = \log \frac{Q_{t,s}}{Q^*} - \frac{1}{\rho} \left(\delta \left(\frac{Q_{t,s}}{Q^*} \eta^* \right) - \delta(\eta^*) \right).$$

This implies that the gap value depends on an investor's beliefs but not her identity, $w_{t,s}^b \equiv w_{t,s}^{i,b}$.

Integrating Eq. (28) forward, we obtain:

$$w_{t,s}^b = \int_t^\infty e^{-\left(\rho + \lambda_s^b\right)(\tilde{t}-t)} \left(W(Q_{\tilde{t},s}) + \lambda_s^b w_{\tilde{t},s'}^b \right) d\tilde{t}. \tag{A.13}$$

Hence, the gap value captures an appropriately discounted present value of instantaneous welfare gaps. Note that $W(Q_{t,s})$ is a strictly concave function maximized at $Q_{t,s} = Q^*$. Therefore, Eq. $(A.13)$ also implies $w_{t,s}^b \le 0$ for each t, s.

A.2.3. Gap value in recession

Next consider the gap value in the recession state $s = 2$. Since the model is stationary, we conjecture that the gap value can be written as a function of optimists' wealth share,

$$w_{t,2}^b = w_2^b(\alpha_{t,s}),$$

for some function $w_2^b(\cdot)$. Differentiating this expression, we have:

$$\frac{\partial w_{t,s}^b}{\partial t} = \frac{dw_2^b(\alpha_{t,s})}{d\alpha} \dot{\alpha}_{t,s}$$

$$= -\frac{dw_2^b(\alpha_{t,s})}{d\alpha} \alpha_{t,2} (1 - \alpha_{t,2}) \Delta\lambda_2^o.$$

Note that $w_{t,3}^b = 0$ since $Q_{t,3} = Q^*$. Finally, recall that we have $Q_{t,2} = Q_2(\alpha) < Q^*$, where $Q_2(\alpha)$ is a strictly increasing function. Substituting these expressions into Eq. (28) for state $s = 2$, we characterize the gap value as the solution to a differential equation in α-domain:

$$\left(\rho + \lambda_2^b\right) w_2^b(\alpha) + \frac{dw_2^b(\alpha)}{d\alpha} \alpha (1 - \alpha) \Delta\lambda_2^o = W(Q_2(\alpha)).$$

We analyze the solution to this differential equation in Caballero and Simsek (2017). In particular, since $W(Q_2(\alpha))$ is strictly increasing in α (since $Q_2(\alpha) < Q^*$), $w_2^b(\alpha)$ is also strictly increasing in α. Using the integral expression in $(A.13)$, we also have $w_2^b(\alpha) < 0$ for each α. This establishes Eq. (29) in the main text.

A.3. Omitted derivations in Section 4

We first characterize the equilibrium for a given leverage limit function, $\bar{\omega}_1(\cdot)$. We then prove Proposition 1, which establishes the comparative statics of tightening the leverage limit (for given α).

To characterize the equilibrium, we assume the parameters satisfy:

Assumption A2. $r_1^{f,p} \equiv \rho + g_1 - \delta(\eta^*) - \lambda_1^p \left(\frac{Q^*}{Q_2^p} - 1 \right) > 0.$

Here, $Q_2^p = Q_2(0) < Q^*$ denotes the asset price in the recession state when pessimists dominate the economy. Assumption A2 ensures that the boom features a positive interest rate even if pessimists dominate. Under this assumption, we conjecture an equilibrium in which the interest rate is positive, $r_{t,1}^f > 0$, and the asset price is at its efficient level, $Q_{t,1} = Q^*$. We also conjecture that the equilibrium outcomes can be described as a function of optimists' wealth share, $\alpha_{t,1}$ (as well as the leverage limit function, $\overline{\omega}_1(\cdot)$). In particular, optimists' wealth share after transition can be written as $\alpha_{t,2} = \alpha_2(\alpha_{t,1}, \overline{\omega}_1)$ (and pessimists' wealth share is the residual, $\alpha_{t,2}^p = 1 - \alpha_{t,2}$).

First consider the corner cases $\alpha_{t,1} = 0$ and $\alpha_{t,1} = 1$. Equivalently, $\alpha_{t,1}^i = 1$ for some belief type i. Using Eq. (21), which holds as equality for type i investors, we obtain:

$$r_1^{f,i} = \rho + g_1 - \delta(\eta^*) - \lambda_1^i \left(\frac{Q^*}{Q_2^i} - 1 \right). \tag{A.14}$$

Under Assumption A2, there exists a solution that satisfies $r_1^{f,i} > 0$ for each $i \in \{o, p\}$. Since $\lambda_1^o < \lambda_1^p$, we also have $r_1^{f,o} > r_1^{f,p}$: the equilibrium interest rate is greater when optimists dominate the economy.

Next consider the intermediate cases, $\alpha_{t,1} \in (0, 1)$. Most of the analysis is in the main text. The remaining step is to show that, when $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha, \infty)$ (when the leverage limit binds) Eq. (34) has a unique solution that satisfies $\alpha_2(\alpha, \overline{\omega}_1) \geq \alpha_2(\alpha, \infty)$. This result follows from Lemma 1 below, which we use in subsequent sections. The lemma applies under the following regularity conditions:

Assumption A3. $Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1 - \alpha_2}$ for $\alpha_2 \in (0, 1)$; and $Q_2\left(\frac{\alpha \lambda_1^o}{\overline{\lambda}_1(\alpha)} \right) > Q^* \alpha \left(1 - \frac{\lambda_1^o}{\lambda_1^p} \right)$ for $\alpha \in (0, 1)$.

These conditions concern the price function in the recession. They are mild, and we can verify that numerical solutions do not violate these conditions. They are also sufficient conditions, i.e., they can be relaxed further. The first part says that the slope of the price function is not too large. Since $Q_2(1) = Q_2^o < Q^*$, this condition will always hold if $Q_2(\alpha_2)$ is a linear function. Therefore, it holds as long as $Q_2(\alpha_2)$ does not deviate from linearity too much. The second part requires that either the price decline after transition to the recession is not too large, or the extent of speculation during the boom is not too large. For instance, when $\alpha = 1$, the requirement is $Q_2^o > Q^* \left(1 - \frac{\lambda_1^o}{\lambda_1^p} \right)$. This holds if Q_2^o is close to Q^* or if λ_1^o is not substantially smaller than λ_1^p.

Lemma 1. *Consider the following function:*

$$f(\alpha_2; \alpha, \overline{\omega}_1) = 1 - \frac{\alpha_2}{\alpha} - (\overline{\omega}_1 - 1) \left[\frac{Q^*}{Q_2(\alpha_2)} - 1 \right],$$

where $\alpha, \overline{\omega}_1$ *are parameters such that* $\alpha \in (0, 1), \overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$. *Under Assumption A3,* $f(\alpha_2) = 0$ *has a unique solution that satisfies* $\alpha_2 \in [\alpha_2(\alpha, \infty), \alpha)$.

Proof. We first show that there exists a solution that lies in the desired interval. We have

$$\begin{aligned} f(\alpha_2(\alpha, \infty)) &= 1 - \frac{\alpha_2(\alpha, \infty)}{\alpha} - (\overline{\omega}_1 - 1) \left[\frac{Q^*}{Q_2(\alpha_2(\alpha, \infty))} - 1 \right] \\ &\geq 1 - \frac{\alpha_2(\alpha, \infty)}{\alpha} - (\omega_1^o(\alpha, \infty) - 1) \left[\frac{Q^*}{Q_2(\alpha_2(\alpha, \infty))} - 1 \right] = 0. \end{aligned}$$

Here, the inequality in the second line follows since $\overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$ and $Q_2(\alpha_2(\alpha, \infty)) < Q^*$, and the

equality follows from the definition of $\omega_1^o(\alpha, \infty)$. We also have

$$f(\alpha) = -(\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right] < 0.$$

It follows that there exists a solution in $[\alpha_2(\alpha, \infty), \alpha)$.

We next show that the derivative of f is strictly negative at each zero of f:

$$f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in [\alpha_2(\alpha, \infty), \alpha) \text{ and } f(\alpha_2) = 0. \tag{A.15}$$

This establishes that f has a unique zero in the desired interval. To establish this claim, we first evaluate the derivative

$$f'(\alpha_2) = -\frac{1}{\alpha} + (\overline{\omega}_1 - 1)\frac{Q^*}{(Q_2(\alpha_2))^2}Q_2'(\alpha_2).$$

Hence, $f'(\alpha_2) < 0$ as long as

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)}\frac{Q_2'(\alpha_2)}{Q_2(\alpha_2)} < 1.$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)} = (\alpha - \alpha_2)\frac{Q^*}{Q^* - Q_2(\alpha_2)}.$$

Combining the last two displayed equations, we need to show

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1 - \alpha_2}\frac{1 - \alpha_2}{\alpha - \alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.16}$$

Using the first part of Assumption A3, we have

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1 - \alpha_2}. \tag{A.17}$$

Using the second part of Assumption A3, we also have

$$1 \leq \frac{1 - \alpha_2(\alpha, \infty)}{\alpha - \alpha_2(\alpha, \infty)}\frac{Q_2(\alpha_2(\alpha, \infty))}{Q^*} \leq \frac{1 - \alpha_2}{\alpha - \alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.18}$$

Here, the first inequality follows from Assumption A3 since $\frac{1 - \alpha_2(\alpha, \infty)}{\alpha - \alpha_2(\alpha, \infty)} = \frac{\lambda_1^p}{\alpha(\lambda_1^p - \lambda_1^o)}$ after substituting $\alpha_2(\alpha, 0) = \frac{\alpha\lambda_1^o}{\overline{\lambda}_1(\alpha)}$ [cf. Eq. (31)]. The second inequality follows since $\alpha_2(\alpha, \infty) \leq \alpha_2$ implies $\frac{1 - \alpha_2(\alpha, \infty)}{\alpha - \alpha_2(\alpha, \infty)} \leq \frac{1 - \alpha_2}{\alpha - \alpha_2}$ and $Q_2(\alpha_2(\alpha, \infty)) \leq Q_2(\alpha_2)$. Combining Eqs. $(A.17)$ and $(A.18)$ establishes Eq. $(A.16)$. This in turn establishes Eq. $(A.15)$ and shows that there is a unique solution. $\qquad\square$

Proof of Proposition 1. Recall that optimists' wealth share after transition corresponds to the zero of the function defined in Lemma 1. Next consider how the solution (characterized in the proof of the lemma) changes with $\overline{\omega}_1$. Implicitly differentiating the equation $f(\alpha_2; \alpha, \overline{\omega}_1) = 0$ with respect to $\overline{\omega}_1$, we obtain:

$$\frac{d\alpha_2}{d\overline{\omega}_1} = \frac{\frac{Q^*}{Q_2(\alpha_2)} - 1}{f'(\alpha_2)} < 0.$$

Here, the inequality follows since $\frac{Q^*}{Q_2(\alpha_2)} - 1 > 0$ and $f'(\alpha_2) < 0$ [cf. Eq. $(A.15)$]. It follows that the

50

solution is strictly decreasing in $\overline{\omega}_1$, that is, $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$. In particular, decreasing the leverage limit increases optimists' wealth share after transition.

To establish the last part, note that Eq. (35) describes optimists' growth absent transition, $\dot{\alpha}_{t,1}/\alpha_{t,1}$, as a *decreasing* function of $\alpha_{t,2}$ (given the parameters and $\alpha_{t,1}$). Combining this observation with $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$, we also find $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d\overline{\omega}_1(\alpha)} < 0$. Hence, decreasing the leverage limit slows down the growth of optimists' wealth share absent transition, completing the proof. □

A.4. Omitted derivations in Section 5

Proof of Proposition 2. Recall that Eq. (23) applies for an arbitrary specification of monetary policy as long as leverage constraints do not bind for either type. When $\overline{\omega}_1 = \infty$, constraints do not bind in state 1. Applying Eq. (23), the evolution of optimists' wealth share is given by (31). In particular, monetary policy does not influence the evolution of optimists' wealth share.

Next note that, using Eq. $(A.13)$, we can write the planner's gap value as

$$w_1^{pl}(\alpha_{0,1}) = \int_0^\infty e^{-(\rho+\lambda_1^{pl})t}\left(W(Q_{t,1}) + \lambda_1^{pl}w_{t,2}^b(\alpha_{t,2})\right)dt.$$

Here, $\alpha_{t,2}$ denotes optimists' wealth share in the recession state if the economy switches to recession at time t. Monetary policy does not affect the path $\{\alpha_{t,2}\}$. Therefore, the previous expression is maximized when $W(Q_{t,1})$ is maximized. This happens when the planner follows the conventional output-stabilization policy and sets $Q_{t,1} = Q^*$. It follows that prudential policy can only lower the gap value function, $w_1^{pl}(\alpha,\infty,Q_1) \le w_1^{pl}(\alpha,\infty)$, completing the proof. □

Proof of Proposition 3. First consider the effect of the leverage limit, $\tilde{\omega}_1$. Since $\tilde{\omega}_1(\alpha) < \omega_1(\alpha,\infty)$, optimists' wealth share, $\alpha_2(\alpha,\tilde{\omega}_1)$, is characterized as the unique solution to the following equation (see Appendix A.3):

$$\frac{\alpha_2(\alpha,\tilde{\omega}_1)}{\alpha} = 1 - (\tilde{\omega}_1(\alpha)-1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right]. \tag{A.19}$$

We will show (constructively) that there exists a PMP that replicates the wealth share. Let $\tilde{\alpha}_2 = \alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right)$ denote optimists' wealth share after transition with PMP. In the conjectured equilibrium, optimists' leverage limit binds (since $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1) > \alpha_2(\alpha,\infty)$). Therefore, optimists' wealth share is the solution to

$$\frac{\tilde{\alpha}_2}{\alpha} = 1 - (\overline{\omega}_1(\alpha)-1)\left[\frac{\tilde{Q}_1(\alpha)}{Q_2(\tilde{\alpha}_2)} - 1\right]. \tag{A.20}$$

We next claim that, for appropriately chosen $\tilde{Q}_1(\alpha)$, this equation holds for $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1)$.

To this end, let $\tilde{Q}_1(\alpha)$ be such that Eq. (37) holds. After rearranging this expression, we can solve for $\tilde{Q}_1(\alpha)$ in closed form:

$$\tilde{Q}_1(\alpha) = Q_2(\alpha_2(\alpha,\tilde{\omega}_1))\left(1 + \frac{\tilde{\omega}_1(\alpha)-1}{\overline{\omega}_1(\alpha)-1}\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right]\right). \tag{A.21}$$

Since $\tilde{\omega}_1(\alpha) < \overline{\omega}_1(\alpha)$, it is easy to check that $\tilde{Q}_1(\alpha) < Q^*$. Since $\tilde{\omega}_1(\alpha) > 1$, we also have $\tilde{Q}_1(\alpha) > Q_2(\alpha_2(\alpha,\tilde{\omega}_1))$. In particular, there exists a unique $\tilde{Q}_1(\alpha) \in (Q_2(\alpha_2(\alpha,\tilde{\omega}_1)),Q^*)$ that satisfies Eq. (37).

We next substitute Eq. (37) into Eq. $(A.19)$, which proves our claim that Eq. $(A.20)$ holds with $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1)$. We can also check that (under Assumption A3) this equation has a unique solution.

This proves $\alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) = \alpha_2\left(\alpha,\tilde{\omega}_1\right)$. Note that Eq. $(A.21)$ implies $\frac{\partial \tilde{Q}_1(\alpha)}{\partial \tilde{\omega}_1(\alpha)} > 0$, which completes the proof of the first part of the proposition .

Next consider the interest rate corresponding to PMP. Since the policy applies only at an infinitesimal instant, it does not affect the price drift, $\dot{Q}_{t,1} = 0$. In particular, the instantaneous return to capital is given by $\tilde{r}_1 = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$ [cf. Eq. (18)]. Combining this with Eq. (21) for pessimists, we obtain the following analogue of Eq. (30):

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p\frac{1-\alpha}{1-\alpha_2(\alpha,\tilde{\omega}_1)}\left(\frac{\tilde{Q}_1(\alpha)}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right).$$

Using Eq. $(A.20)$ to substitute for the price decline, we can rewrite this as

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p\frac{1-\alpha}{\alpha}\frac{\alpha-\alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)}\frac{1}{\overline{\omega}_1(\alpha)-1}. \tag{A.22}$$

Absent prudential policy, the interest rate is characterized by Eq. (30). After substituting for the price decline from (20), we can rewrite this expression as

$$r_1^f(\alpha,\overline{\omega}_1) = \rho + g_1 - \delta(\eta^*) - \lambda_1^p\frac{1-\alpha}{\alpha}\frac{\alpha_2(\alpha,\overline{\omega}_1)-\alpha}{1-\alpha_2(\alpha,\overline{\omega}_1)}\frac{1}{\omega_1(\alpha,\overline{\omega}_1)-1}. \tag{A.23}$$

Here, $\omega_1(\alpha,\overline{\omega}_1)$ denotes the equilibrium leverage ratio.

Next note that $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) < \delta(\eta^*)$ since $\tilde{Q}_1(\alpha) < Q^*$. Note also that $\frac{\alpha-\alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)} < \frac{\alpha-\alpha_2(\alpha,\overline{\omega}_1)}{1-\alpha_2(\alpha,\overline{\omega}_1)}$ since $\alpha_2(\alpha,\tilde{\omega}_1) > \alpha_2(\alpha,\overline{\omega}_1)$. Finally, note that $\frac{1}{\overline{\omega}_1(\alpha)-1} \leq \frac{1}{\omega_1(\alpha,\overline{\omega}_1)-1}$ since $\omega_1(\alpha,\overline{\omega}_1) \leq \overline{\omega}_1(\alpha)$. Combining these observations with Eqs. $(A.22)$ and $(A.23)$ proves that $\tilde{r}_1^f = r_1^f\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f(\alpha,\overline{\omega}_1)$: PMP raises the interest rate.

Finally, consider how raising the leverage limit $\tilde{\omega}_1(\alpha)$ affects the interest rate with PMP. Since raising the leverage limit increases $\tilde{Q}_1(\alpha)$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces $\alpha_2(\alpha,\tilde{\omega}_1)$, it also increases the term $\frac{\alpha-\alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)}$. Combining these observations with $(A.22)$ proves that raising the leverage limit decreases $\tilde{r}_1$, that is: $\frac{\partial r_1^f(\alpha,\overline{\omega}_1,\tilde{Q}_1)}{\partial \tilde{\omega}_1(\alpha)} < 0$. In particular, targeting a lower effective leverage limit $\tilde{\omega}_1(\alpha)$ requires a higher interest rate, completing the proof. $\qquad\square$

Proof of Proposition 4. We have the following closed-form solution for the price function:

$$Q_1\left(\alpha,\hat{l}\right) = \begin{cases} Q^* & \text{if } \omega_1\left(\alpha,\hat{l}\right) < \overline{\omega}_1\left(\alpha,\hat{l}\right) \\ Q_2\left(\alpha_2\left(\alpha,\hat{l}\right)\right)\left(1 + \frac{\overline{\omega}_1(\alpha,\hat{l})-1}{\overline{\omega}_1(\alpha,\hat{l})-1}\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\hat{l}))} - 1\right]\right) < Q & \text{if } \omega_1\left(\alpha,\hat{l}\right) = \overline{\omega}_1\left(\alpha,\hat{l}\right) \end{cases}. \tag{A.24}$$

Here, the first line corresponds to the case in which the leverage limit does not bind under $\hat{l}$. In this case, the monetary authority does not use PMP. The second line corresponds to the case in which the leverage limit binds. In this case, the monetary authority uses PMP. Moreover, using Eq. $(A.21)$ we have a closed-form solution for the asset price level.

One difference from Proposition 3 concerns the characterization of the interest rate. Since the policy is applied dynamically, the price drift, $\dot{Q}_{t,1}$, is not necessarily zero, which affects the level of the interest

rate. To characterize this effect, note that:

$$\dot{Q}_{t,1} = \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\dot{\alpha}_{t,1}$$

$$= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha_{t,1}\frac{1-\alpha_{t,1}}{1-\alpha_{t,2}}\left(\alpha_{t,1}-\alpha_{t,2}\right)$$

$$= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha\left(1-\alpha\right)\frac{\alpha-\alpha_2\left(\alpha,\tilde{l}\right)}{1-\alpha_2\left(\alpha,\tilde{l}\right)}. \tag{A.25}$$

Here, the second line substitutes the evolution of optimists' wealth share from Eq. (22) and the third line substitutes $\alpha_{t,1}=\alpha$ and $\alpha_{t,2}=\alpha_2\left(\alpha,\tilde{l}\right)$. The expression $\frac{\partial Q_1(\alpha,\tilde{l})}{\partial_+\alpha}$ corresponds to the right-derivative of the function characterized in $(A.24)$.[10] We can check that the right-derivative, $\frac{\partial Q_1(\alpha,\tilde{l})}{\partial_+\alpha}$, is continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$ (because $Q_1\left(\alpha,l\right)=Q^*$ for each α). Consequently, when viewed as a function of $\tilde{l}$, the price drift, $\dot{Q}_{t,1}$, is also continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$.

Next note that, following similar steps as in the proof of Proposition 4, the interest rate in this case can be written as

$$\tilde{r}_1^f = \rho + g_1 + \dot{Q}_{t,1} - \delta\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q^*}\eta^*\right) - \lambda_1^p\frac{1-\alpha}{1-\alpha_2\left(\alpha,\tilde{l}\right)}\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q_2\left(\alpha_2\left(\alpha,\tilde{l}\right)\right)}-1\right),$$

where $\dot{Q}_{t,1}$ is given by Eq. $(A.25)$. When viewed as a function of $\tilde{l}$, the interest rate $\tilde{r}_1^f$ is continuous in $\tilde{l}$, and it is equal to the benchmark interest rate $r_1^f\left(\alpha,l\right)$ when $\tilde{l}=l$. Recall that the benchmark rate is strictly positive for each $\alpha\in\left(0,1\right)$ [cf. Section 4]. Therefore, $\tilde{r}_1^f>0$ for each $\alpha\in\left(0,1\right)$ as long as $\tilde{l}$ is in a sufficiently small neighborhood of l. In particular, PMP doesn't violate the zero lower bound constraint on the interest rate.

Next consider the second part. Using Eq. $(A.13)$, we can write the planner's gap value with policy $\tilde{l}$ as

$$w_1^{pl}\left(\alpha_{0,1},\tilde{l}\right) = \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q^*\right)+\lambda_1^{pl}w_{t,2}^b\left(\alpha_{t,2}\right)\right)dt. \tag{A.26}$$

Here, $\alpha_{t,2}$ denotes optimists' wealth share if the economy transitions to recession at time t. Likewise, we write the planner's gap value with policy $Q_1\left(\alpha,\tilde{l}\right)$ as

$$w_1^{pl}\left(\alpha_{0,1},Q_1\left(\alpha,\tilde{l}\right)\right) = \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha,\tilde{l}\right)\right)+\lambda_1^{pl}w_{t,2}^b\left(\alpha_{t,2}\right)\right)dt. \tag{A.27}$$

Next note that, using the first part of this proposition, optimists' wealth share after transition, $\alpha_{t,2}$ (conditional on $\alpha_{t,1}$), is the same under policies $\tilde{l}$ and $Q_1\left(\cdot,\tilde{l}\right)$. Combining this observation with Eq. (22), we also find that the evolution of optimists' wealth share absent transition, $\dot{\alpha}_{t,1}/\alpha_{t,1}$, is the same under both policies. Consequently, optimists' wealth share follows an identical path under both policies. In view of this observation, after taking the difference of Eqs. $(A.27)$ and $(A.26)$, we obtain Eq. (40) in the main text.

---

[10]Note that the function is piecewise differentiable so the right-derivative always exists. The equation depends on the right-derivative (as opposed to left) because $\dot{\alpha}_{t,1}>0$, so $\alpha_{t,1}$ grows over time.

Next note that Eq. $(A.24)$ implies (for a given $\alpha \in (0,1)$) that the prudential asset price level is differentiable in $\tilde{l}$ with a finite derivative. Note also that $Q_1(\alpha, l) = Q^*$. Therefore, taking the derivative of Eq. (40) with respect to $\tilde{l}$ and evaluating at $\tilde{l} = l^*$, we obtain:

$$
\begin{aligned}
\frac{dw_1^{pl}\left(\alpha, l, Q_1\left(\alpha, \hat{l}\right)\right)}{d\tilde{l}}\Big|_{\tilde{l}=l} - \frac{dw_1^{pl}\left(\alpha, \hat{l}\right)}{d\tilde{l}}\Big|_{\tilde{l}=l} &= \int_0^\infty e^{-\left(\rho + \lambda_1^{pl}\right)t} \frac{dW\left(Q^*\right)}{dQ_{t,1}} \frac{dQ_1\left(\alpha_{t,1}, \hat{l}\right)}{d\tilde{l}}\Big|_{\tilde{l}=l} dt \\
&= 0.
\end{aligned}
$$

Here, the first line applies the chain rule and the second line uses the observation that $\frac{dW(Q^*)}{dQ_{t,1}} = 0$ [cf. Eq. (28)]. Rearranging this expression establishes Eq. (39) and completes the proof. $\quad\square$

A.5. Omitted derivations in Section 7

We first state and prove a generalization of Lemma 1, which implies that Eq. (48) has a unique solution (when $Q_1 = Q^*$). We then prove Propositions 5 and 6.

Lemma 2. *Consider the following function:*

$$
f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right) = 1 - \left(\overline{\omega}_1 - 1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\right)} - 1\right] - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right),
$$

where $\alpha, \beta, \overline{\omega}_1$ are parameters such that $\alpha, \beta \in (0,1), \overline{\omega}_1 \leq \omega_1^o(\alpha, \beta, \infty)$. Under Assumption A3, $f(\alpha_2) = 0$ has a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$.

Proof. Following similar steps as in Lemma 1, it is easy to check that $f(\alpha_2(\alpha, \infty)) > 0$ and $f(\alpha) < 0$. This establishes that there exists a solution that lies in the desired interval, $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$.

We next show that the derivative of f is strictly negative at each zero of f, that is:

$$
f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in (\alpha_2(\alpha, \infty), \alpha) \text{ and } f(\alpha_2) = 0.
$$

This establishes that f has a unique zero in the desired interval. To prove the claim, we first evaluate the derivative

$$
f'(\alpha_2) = \left(\overline{\omega}_1 - 1\right)\frac{Q^*}{\left(Q_2\left(\alpha_2\right)\right)^2}Q_2'(\alpha_2) - \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).
$$

Hence, $f'(\alpha_2) < 0$ as long as

$$
\frac{\overline{\omega}_1 - 1}{Q_2\left(\alpha_2\right)}\frac{Q^*}{Q_2\left(\alpha_2\right)}Q_2'(\alpha_2) < \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).
$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies:

$$
\frac{\overline{\omega}_1 - 1}{Q_2\left(\alpha_2\right)} = \frac{1}{Q^* - Q_2\left(\alpha_2\right)}\left(1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right).
$$

Combining the last two displayed equations, we need to show

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}\frac{Q_2(\alpha_2)}{Q^*}g(\alpha_2,\alpha,\beta)$$

$$\text{where } g(\alpha_2,\alpha,\beta) = \frac{(1-\alpha_2)\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)}{1-\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)}.$$

Note that, in the proof of Lemma 1, we already established this inequality for $\beta = 0$ (under Assumption A3). Hence, it suffices to show that $g(\alpha_2,\alpha,\beta) \geq g(\alpha_2,\alpha,0)$. This inequality holds because,

$$g(\alpha_2,\alpha,\beta) > \frac{1-\alpha_2}{1-\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)} > \frac{1-\alpha_2}{1-\frac{\alpha_2}{\alpha}} = g(\alpha_2,\alpha,0).$$

Here, the first inequality follows because $\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right) < \frac{1}{\alpha}$. The second inequality follows because $\alpha_2 > \alpha_2(\alpha,\infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o+(1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$, which in turn implies $\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) > \frac{\alpha_2}{\alpha}$. This establishes the claim and completes the proof of the lemma. $\square$

Proof of Proposition 5. First consider the evolution of α and β absent transition to recession. Applying (22) for regulated and unconstrained optimists (in state $s = 1$), we obtain:

$$\frac{d(\alpha(1-\beta))/dt}{\alpha(1-\beta)} = \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right) \tag{A.28}$$

$$\frac{d(\alpha\beta)/dt}{\alpha\beta} = \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2\beta_2}{\alpha\beta}\right)$$

Solving these equations for $\dot{\alpha}$ and $\dot{\beta}$, we obtain Eq. (43) in the main text.

Next consider the characterization of α_2. In the main text, we established that Eq. (48) characterizes α_2. Lemma 2 implies that there exists a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha,\infty),\alpha)$. Combining this with Eq. (43) also implies $\dot{\alpha} > 0$.

Next note that Eq. (44) characterizes β_2 conditional on α_2. Note also that $\alpha_2 > \alpha_2(\alpha,\infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o+(1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$. Combining this with Eq. (44), we obtain $\frac{\beta_2}{\beta} = \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha/(1-\alpha)}{\alpha_2/(1-\alpha_2)} < 1$. This proves $\beta_2 < \beta$. Combining this with Eq. (43) also implies $\dot{\beta} > 0$.

Next consider the comparative statics of α_2 with respect to β. Recall that α_2 is characterized as the unique solution to the equation, $f(\alpha_2;\alpha,\beta,\overline{\omega}_1) = 0$, where $f(\cdot)$ is defined in Lemma 2. Implicitly differentiating the equation with respect to β, we obtain:

$$\frac{d\alpha_2}{d\beta} = \frac{\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\beta}{-\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\alpha_2},$$

where the derivatives are evaluated at the solution. From the proof of Lemma 2, we also know that $f'(\alpha_2;\alpha,\beta,\overline{\omega}_1) < 0$ when $f(\alpha_2) = 0$. Hence, $\frac{d\alpha_2}{d\beta} < 0$ as long as $\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\beta < 0$. The latter

inequality holds because:

$$
\begin{aligned}
\frac{\partial f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right)}{\partial \beta} &= -\frac{\partial}{\partial \beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right) \\
&= -\frac{\partial}{\partial \beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right) + \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right) \\
&= -\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\left(\frac{\partial}{\partial \beta}\frac{1}{1-\beta}\right) < 0.
\end{aligned}
$$

Here, the last inequality follows since $\frac{\alpha_2}{\alpha} > \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ (since $\alpha_2 > \alpha_2(\alpha, \infty)$) and $\frac{\partial}{\partial \beta}\frac{1}{1-\beta} > 0$. This proves $\frac{d\alpha_2}{d\beta} < 0$.

Next consider the limit as $\beta \to 1$. For any $\alpha_2 > \alpha_2(\alpha, \infty)$, we have

$$
\begin{aligned}
\lim_{\beta \to 1} f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right) &= \lim_{\beta \to 1}\left[\begin{array}{c} 1 - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right] + \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha} \\ -\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right) \end{array}\right] \\
&= -\infty
\end{aligned}
$$

Here, the last line follows because $\frac{\alpha_2}{\alpha} > \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ and $\lim_{\beta \to 1}\frac{-1}{1-\beta} = -\infty$. This also implies $\lim_{\beta \to 1}\alpha_2 = \alpha_2(\alpha, \infty)$ because α_2 is characterized as the unique solution to the equation, $f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right) = 0$, over the range $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$.

Next consider the comparative statics with respect to the leverage limit, $\overline{\omega}_1 = \overline{\omega}_1(\alpha, \beta)$. Following similar steps, we obtain:

$$
\begin{aligned}
\frac{d\alpha_2}{d\overline{\omega}_1} &= \frac{-\partial f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right)/\partial \overline{\omega}_1}{\partial f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right)/\partial \alpha_2} \\
&= \frac{\frac{Q^*}{Q_2(\alpha_2)} - 1}{(\overline{\omega}_1 - 1)\frac{Q^*}{(Q_2(\alpha_2))^2}Q_2'(\alpha_2) - \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)} < 0.
\end{aligned}
$$

Here, the first equality evaluates the partial derivatives and the second inequality uses $\partial f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right)/\partial \alpha_2 < 0$.

Finally, consider the sign of the cross-partial derivative $\frac{\partial}{\partial \beta}\frac{d\alpha_2}{d\overline{\omega}_1}$. We have

$$
\begin{aligned}
sign\left(\frac{\partial}{\partial \beta}\frac{d\alpha_2}{d\overline{\omega}_1}\right) &= sign\left(\frac{\partial}{\partial \beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)\right) \\
&= sign\left(\frac{\partial}{\partial \beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right) - \frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right) \\
&= sign\left(\frac{\partial}{\partial \beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)\right) > 0.
\end{aligned}
$$

This proves $\frac{\partial}{\partial \beta}\frac{d\alpha_2}{d\overline{\omega}_1} > 0$ and completes the proof. $\square$

Proof of Proposition 6. The proof follows similar steps to Proposition 3. Using Eq. 48, α_2 corresponding to the alternative leverage limit is characterized as the unique solution to:

$$
\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) = 1 - (\tilde{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right]. \tag{A.29}
$$

Likewise, α_2 corresponding to the PMP (with the current leverage limit) is characterized as the solution to:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) = 1 - (\overline{\omega}_1 - 1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)} - 1\right]. \tag{A.30}$$

Next note that the proof of Proposition 3 establishes that there is a unique level of $\tilde{Q}_1$ that ensures Eq. (37) holds. Let $\tilde{Q}_1$ denote this level, that is:

$$(\overline{\omega}_1 - 1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)} - 1\right] = (\tilde{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right]. \tag{A.31}$$

Substituting $\tilde{Q}_1$ into Eq. $(A.30)$ ensures that this equation is the same as Eq. $(A.29)$. Therefore, the solutions are the same. Hence, there exists a PMP that replicates α_2 that results from the alternative leverage limit. Recall also that β_2 is characterized by Eq. (44) conditional on α_2. Thus, the same PMP also replicates β_2 that results from the alternative leverage limit. Note also that Eq. $(A.31)$ implies $\frac{\partial \tilde{Q}_1}{\partial \tilde{\omega}_1} > 0$. This completes the proof of the first part.

Next consider the interest rate corresponding to PMP. Note that Eq. (21) continues to hold as equality for pessimists. This implies that the interest rate is given by:

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right) - \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(\frac{\tilde{Q}_1}{Q_2(\alpha_2)} - 1\right).$$

Using Eq. (45) (that describes the wealth share after transition for regulated optimists) to substitute for the price decline, we obtain:

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right) - \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)\frac{1}{\overline{\omega}_1 - 1}. \tag{A.32}$$

For the benchmark without any prudential policy, following similar steps we obtain:

$$r_1^{f,b} = \rho + g_1 - \delta(\eta^*) - \lambda_1^p\frac{1-\alpha}{1-\alpha_2^b}\left(1 - \frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha(1-\beta)}\right)\frac{1}{\omega_1^b - 1}. \tag{A.33}$$

Here, $\alpha_2^b, \beta_2^b, \omega_1^b$ denote the equilibrium variables in the benchmark, which are potentially different than the equilibrium with PMP. In particular, recall from Proposition 5 that $\alpha_2 > \alpha_2^b$. Combining this with Eq. (44), we further obtain $\beta_2 < \beta_2^b$. PMP decreases the fraction of optimists' wealth held by unconstrained optimists, because they react to the policy by increasing their risks more than regulated optimists.

Next note that the wealth shares satisfy the following identity:

$$\begin{aligned}\frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right) &= (1-\alpha)\frac{1 - \frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}}{1-\alpha_2} \\ &= (1-\alpha)\left(1 - \frac{\alpha_2}{1-\alpha_2}\left[\frac{1-\beta_2}{\alpha(1-\beta)} - 1\right]\right). \end{aligned} \tag{A.34}$$

Here, the term in the brackets is positive because $\beta_2 < \beta$. This identity holds for the pair, (α_2, β_2), as well as the pair, $\left(\alpha_2^b, \beta_2^b\right)$. Combining the identity with the inequalities, $\alpha_2 > \alpha_2^b$ and $\beta_2 < \beta_2^b$ (which

implies $1 - \beta_2 > 1 - \beta_2^b$), we further obtain:

$$\frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2\left(1-\beta_2\right)}{\alpha\left(1-\beta\right)}\right) < \frac{1-\alpha}{1-\alpha_2^b}\left(1 - \frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha\left(1-\beta\right)}\right).$$

Note also that $\frac{1}{\overline{\omega}_1 - 1} \leq \frac{1}{\omega_1^b - 1}$ since $\omega_1^b \leq \overline{\omega}_1$. Finally, we also have $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right) < \delta\left(\eta^*\right)$ since $\tilde{Q}_1 < Q^*$. Combining these inequalities with Eqs. $(A.32)$ and $(A.33)$ proves that $\tilde{r}_1^f > r_1^{f,b}$: that is, PMP sets a higher interest rate than in the benchmark without prudential policies.

Finally, consider how raising the target leverage limit $\tilde{\omega}_1$ affects the interest rate corresponding to PMP. Since raising the leverage limit increases $\tilde{Q}_1$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces α_2, it also increases β_2 (and reduces $1-\beta_2$). Combining this with the identity in $(A.34)$ implies that raising the leverage limit raises the term, $\frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)$. Combining these observations with $(A.32)$ proves that raising the target leverage limit decreases the interest rate, that is: $\frac{\partial \tilde{r}_1^f}{\partial \tilde{\omega}_1} < 0$. In particular, targeting a lower effective leverage limit $\tilde{\omega}_1$ requires setting a higher interest rate, completing the proof. $\square$

A.6. Details of the numerical exercise in Sections 5 and 6

For depreciation, we use the functional form

$$\delta\left(\eta\right) = \underline{\delta} + \left(\overline{\delta} - \underline{\delta}\right)\frac{\left(\eta - \underline{\eta}\right)^{1+1/\varepsilon}}{1 + 1/\varepsilon} \quad \text{for } \eta \geq \underline{\eta} \tag{A.35}$$

(and $\delta\left(\eta\right) = \underline{\delta}$ for $\eta < \underline{\eta}$) given some constants $\underline{\delta}, \left(\overline{\delta} - \underline{\delta}\right), \underline{\eta}, \varepsilon > 0$. This functional form implies that decreasing factor utilization below the efficient level, η^*, reduces the depreciation rate until $\underline{\eta} < \eta^*$, but it has no effect on depreciation beyond this level. Raising factor utilization above the efficient level increases capital depreciation.

In our numerical examples, we set

$$\underline{\eta} = 0.97, \ \underline{\delta} = 0.04, \overline{\delta} = 0.087, \varepsilon = 20.$$

These choices ensure that the efficient factor utilization and the corresponding depreciation rate are given by [cf. Eq. (16)] with

$$\eta^* = 1 \text{ and } \delta\left(\eta^*\right) = 0.041.$$

In particular, we normalize the efficient factor utilization to one. The choice of $\underline{\eta} = 0.97$ (together with a relatively high elasticity, $\varepsilon = 20$) implies that underutilizing capital by up to 3 percent is not too costly, since it is compensated by a relatively large decline in depreciation. Underutilizing capital beyond this level is much costlier as there is no compensation in terms of reduced depreciation. In our examples, this means that underutilizing capital in the recession is much costlier than underutilizing capital during the boom.

For the discount rate, we set

$$\rho = 0.04.$$

This choice (together with the specification for the depreciation function) ensures that Assumption 2

holds. For the productivity level, we set $A = 1$. This does not play a role as it scales all variables. These choices imply that the efficient asset price level is given by [cf. Eq. (17)]:

$$Q^* = \frac{A\eta^*}{\rho} = 25.$$

For the productivity *growth* rates, we set

$$g_3 = g_1 = 0.1 - (\rho - \delta(\eta^*)) = 0.101$$
$$g_2 = -0.05 - (\rho - \delta(\eta^*)) = -0.049.$$

These choices satisfy $g_2 < \min(g_1, g_3)$. They also imply that, with no state changes or belief disagreements and if capital were perfectly utilized, then the (risk-adjusted) return to capital would be equal to 10% in the boom and the recovery states and -5% in the recession state [cf. (18)]. In particular, the transition from the boom to the recession represents a 15% shock to asset valuations.

For beliefs, we set

$$\lambda_1^o = 0.09 \text{ and } \lambda_1^p = 0.9$$
$$\lambda_2^o = 4.97 \text{ and } \lambda_2^p = 0.49$$

(and $\lambda_3^o = \lambda_3^p = 0$). These beliefs satisfy Assumption 1: compared to pessimists, optimists assign a smaller probability to a transition from boom to recession but a greater probability to a transition from recession to recovery. When combined with the remaining parameters, these values satisfy Assumptions A1-A2, the regularity conditions we need in order to obtain an equilibrium with a positive interest rate in the boom state and a zero interest rate (and suboptimal asset price level) in the recession state.

Recall that we also need Assumption A3 (which is a regularity condition) to ensure that there is a unique equilibrium when optimists' leverage limit binds (cf. Appendix A.3). This condition depends on the equilibrium price function in the recession, $Q_2(\alpha)$. Figure 9 plots the price function corresponding to the parameters described above. We verify that, when combined with the remaining parameters, this function satisfies Assumption A3.