Concerted Efforts?
Monetary Policy and Macro-Prudential Tools*

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May 9, 2017

Abstract

The inception of macro-prudential policy frameworks in the wake of the global financial crisis raises questions of how new policy tools should be operated and how macro-prudential and monetary policies should be coordinated. We examine these questions through the lens of a macroeconomic model featuring nominal rigidities, housing, incomplete risk-sharing between borrower and saver households, and macro-prudential mortgage loan-to-value and bank capital tools. We derive a welfare-based loss function which suggests a role for active macro-prudential policy to enhance risk-sharing. Macro-prudential policy faces tradeoffs, however, and complete macro-prudential stabilisation is not generally possible in our model. Nonetheless, simulations of a housing boom and bust suggest macro-prudential tools could alleviate debt-deleveraging and help avoid zero lower bound episodes, even when macro-prudential tools themselves impose only occasionally binding constraints on debt dynamics in the economy.

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1 Introduction

> *With the recovery in the UK economy broadening and gaining momentum in recent months, the Bank of England is now focussed on turning that recovery into a durable expansion. To do so, our policy tools must be used in concert.* Carney (2014)

In the aftermath of the global financial crisis, macro-prudential policy frameworks have been established and developed across the world. The inception of these frameworks raises questions of how new policy tools should be operated and how macro-prudential and monetary policies should be coordinated.

These questions have particular resonance given the conditions currently facing policymakers in many advanced economies. Since the financial crisis, monetary policy has been set with a view to supporting economic activity and hence preventing inflation from falling below target. That has required a prolonged period of low, sometimes negative, short-term policy rates and a raft of so-called unconventional monetary policy measures. These policies have supported asset prices and kept borrowing costs low. However, these effects have also given rise to concerns that such monetary conditions may lead to levels of indebtedness that threaten financial stability. In some cases, macro-prudential policy instruments have been used to guard against these risks. So the policy mix in many economies has been 'loose' monetary policy and 'tight' macro-prudential policy.[1]

In this paper, we examine these questions through the lens of a simple and commonly used modelling framework. Our model is rich enough to generate meaningful policy tradeoffs, but sufficiently simple to deliver tractable expressions for welfare and analytical results under some parameterisations.

We work with a simple model incorporating borrowing constraints and nominal rigidities. These frictions give rise to meaningful roles for macro-prudential policy and monetary policy respectively. The financial friction takes the form of a collateral constraint, following Kiyotaki and Moore (1997) and Iacoviello (2005) among many others. The collateral constraint limits the amount that relatively impatient households can borrow. Specifically, their debt cannot exceed a particular fraction of the value of the housing stock that they own: there is a 'loan to value' (LTV) constraint which the macro-prudential authority can vary. In turn, borrowing by relatively impatient households is financed by saving by more patient households ('savers'). A perfectly competitive banking system intermediates the flow of saving from savers to borrowers. Banks are subject to a capital requirement and we assume that raising equity is costly (see also Justiniano et al., 2014). As such, variation in capital requirements generates fluctuations in the spread between borrowing and deposit rates,

[1] The quote at the start of the paper is from the opening statement at a press conference explaining the decision of Bank of England's Financial Policy Committee to limit the quantity of new lending at high loan-to-value ratios. That statement explains that: "The existence of macro-prudential tools allows monetary policy to focus on its primary responsibility of price stability. In other words, monetary policy does not need to be diverted to address a sector-specific risk in the housing market." Similarly, authorities in Canada have tightened macro-prudential policy several times since the financial crisis (see Krznar and Morsink, 2014) while the official policy rate has remained low. Conversely, the policy mix in Sweden has been a subject of much controversy and debate (see, for example, Jansson, 2014; Svensson, 2011).

providing the authorities with a second macro-prudential tool in addition to the LTV, albeit one that is costly to deploy. Finally, the nominal frictions are Calvo (1983) contracts, standard in the New Keynesian literature.

We derive a welfare-based loss function as a quadratic approximation to a weighted average of the utilities of borrowers and savers. The loss function has five (quadratic) terms. Two terms stem from the nominal rigidities in the model and are familiar from New Keynesian models: the policymaker seeks to stabilise the output gap and inflation. The remaining terms are generated by the financial friction: the policymaker seeks to stabilise the distribution of non-durable consumption and housing consumption between borrowers and savers—the 'consumption gap' and the 'housing gap' respectively. The presence of household heterogeneity therefore gives rise to objectives whose origin lies in the incompleteness of risk sharing between households in the economy. The final term in the loss function captures the costs of varying capital requirements, which themselves stem from the non-zero cost of equity we assume outside of steady state.

We use the model to study how monetary and macro-prudential policies should optimally respond to shocks. To build some intuition, we first focus on a linear approximation of the model around a steady state in which the borrowing constraint is always binding and the value of housing can be fully used as collateral. We demonstrate that macro-prudential policy generally faces a tradeoff in stabilising the distribution of consumption and the distribution of housing services even when prices are flexible and both macro-prudential tools are available to use. We also show that monetary policy alone has relatively little control over these distributions, particularly the distribution of housing between borrowers and savers. In other words, imperfect risk sharing is a real phenomenon whose consequences could be addressed by macro-prudential policies, but these policies also imply costs that must be accounted for in deploying them. This tradeoff prevents complete macro-prudential stabilisation given the tools we study even under flexible prices.

To examine the quantitative implications of the model for optimal monetary and macro-prudential policy, we then present some numerical experiments designed to simulate a housing boom and bust calibrated with reference to that experienced in the United Kingdom and the United States in the decades preceding and following 2008 (Figure 1). This allows use to examine to what extent macro-prudential tools could have complemented monetary policy in achieving macroeconomic stabilisation goals in the face of a housing boom and bust which forces the nominal rate to the zero bound. We find that macro-prudential policies, in the form of the LTV tool and bank capital requirements, allow for better stabilisation of the consumption and housing gaps, but also allow monetary policy to fully stabilise the output gap and inflation because the short-term nominal interest rate does not hit the zero bound. In other words, the optimal conduct of macro-prudential policies leads to an increase in the natural rate of interest.

Some of our simulations have the property that the LTV tool may be only occasionally effective. If the expected evolution of house prices is such that the borrowing constraint becomes slack, then the policymaker may be unable to engineer an allocation in which the housing gap is closed. This suggests that macro-prudential policy tools may be less than fully effective during periods

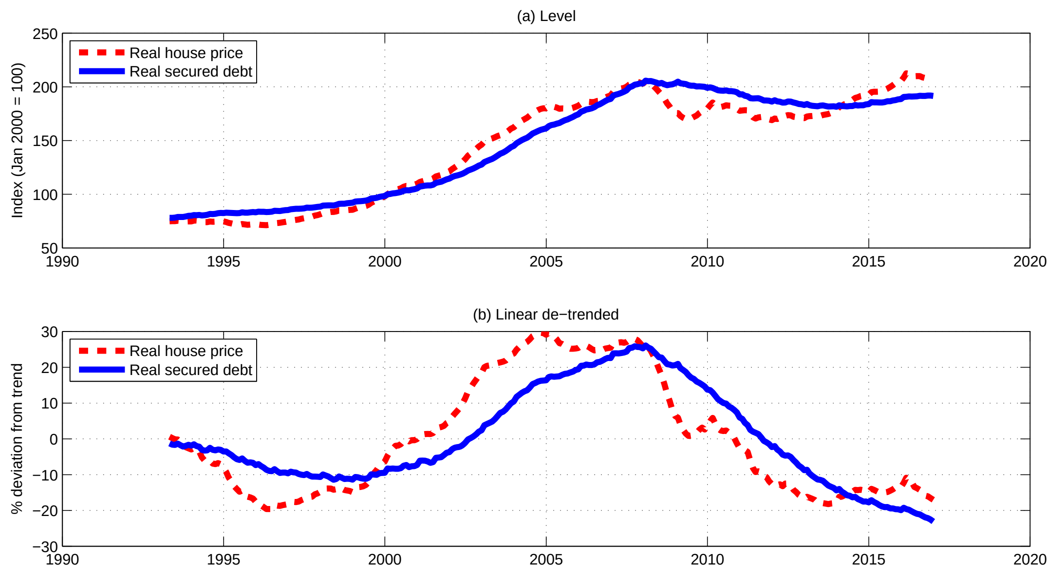

Figure 1: Boom and bust in UK house prices and mortgage debt

of sustained house price increases, such as those observed in many economies before the financial crisis and illustrated for the UK in Figure 1.

We contribute to a growing literature exploring the conduct and coordination of macro-prudential policy. In the context of optimal policy, Angelini et al. (2012), Bean et al. (2010) and De Paoli and Paustian (2013) consider the coordination between monetary and macro-prudential policies in models with similar frictions to ours. Those papers also find that there are cases in which monetary and macro-prudential policy tools may be moved in opposite directions. Most of those models are more complex than ours and consequently the loss functions of the policymaker(s) are not based on the welfare of agents in the models. The welfare-based loss function in our model is similar to that derived by Andres et al. (2013) in a similar model, and bears similarity to Curdia and Woodford (2010). However, those authors focus on the analysis of optimal monetary policy and do not explore macro-prudential policy.

Other papers with a greater focus on macro-prudential policies include Clerc et al. (2015), Angeloni and Faia (2013), Gertler et al. (2012) and Christiano and Ikeda (2016). The focus of each of these papers is on macro-prudential bank capital instruments, whereas we also consider a macro-prudential LTV tool. The LTV instrument is empirically relevant in particular for countries like the UK, where mortgage lending is the single largest asset class on domestic banks' balance sheets, and is also the single largest liability class on households' balance sheets.

Finally, Eggertsson and Krugman (2012) study the implications of households' debt-deleveraging for monetary policy in a model similar to ours, while Korinek and Simsek (2016) and Farhi and Werning (2016) study the theoretical implications of the zero bound constraint on monetary policy and distortions in financial markets for optimal macro-prudential policies.

The rest of the paper is organised as follows. Section 2 presents the model. Section 3 derives a linear-quadratic approximation of the equilibrium and discusses some analytical results. Section 4 illustrates the optimal joint conduct of monetary and macro-prudential policy via a numerical simulation. Section 5 concludes.

2 Model

The economic agents in the model are households, banks, firms, and the government. Households are heterogeneous in their degree of patience. Banks transfer funds from savers to borrowers and fund their operations with a mix of deposits and equity. Firms produce goods for consumption. The government conducts monetary and macro-prudential policy.

2.1 Households

Patient households (i.e. savers, indexed by s) have a higher discount factor than impatient households (i.e. borrowers, indexed by b). We denote with $\xi \in (0,1)$ the mass of borrowers, and normalise the total size of the population to one. We also assume perfect risk sharing within each group.

2.1.1 Savers

A generic saver household $i \in [0, 1-\xi)$ decides how much to consume in goods $C_t^s(i)$ and housing services $H_t^s(i)$, save in deposits $D_t^s(i)$ and equity $E_t^s(i)$ of financial intermediaries, and work $L_t^s(i)$, to maximise

$$\mathbb{W}_0^s(i) \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta_s^t \left[\left(1 - e^{-zC_t^s(i)}\right) + \frac{\chi_H^s}{1-\sigma_h} (H_t^s(i))^{1-\sigma_h} - \frac{\chi_L^s}{1+\varphi} (L_t^s(i))^{1+\varphi} \right] \right\}, \tag{1}$$

where $\beta_s \in (0,1)$ is the individual discount factor, $z > 0$ is the degree of absolute risk aversion, $\sigma_h \geq 0$ is the inverse elasticity of housing demand, $\varphi \geq 0$ is the inverse Frisch elasticity of labour supply, and χ_H^s, $\chi_L^s > 0$ are type-specific normalisation constants.

The budget constraint for patient household i is

$$P_t C_t^s(i) + D_t^s(i) + E_t^s(i) + (1+\tau^h) Q_t H_t^s(i) = \\ W_t^s L_t^s(i) + R_{t-1}^d D_{t-1}^s(i) + R_{t-1}^e E_{t-1}^s(i) + Q_t H_{t-1}^s(i) - T_t^s(i) + \Omega_t^s(i) - \Gamma_t(i),$$

where P_t is the consumption price index, Q_t is the nominal house price, W_t^s is the nominal wage for savers, R_{t-1}^d is the nominal return on bank deposits, and R_{t-1}^e is the nominal return on bank equity.[2] The variable $T_t^s(i)$ captures lump-sum taxes while $\Omega_t^s(i)$ denotes the savers' share of remunerated profits from intermediate goods producers and from banks. The constant τ^h is a tax/subsidy on savers' housing that we later set to deliver an efficient steady state in the housing market. The final

[2] As in Benigno et al. (2014), the introduction of type-specific wages and exponential utility simplifies aggregation, and facilitates the derivation of a welfare criterion for the economy as a whole.

term in the budget constraint is a cost associated with deviations from some preferred portfolio level of bank equity $\tilde{\kappa} > 0$

$$\Gamma_t(i) \equiv \frac{\Psi}{2}\left[\frac{E_t^s(i)}{\tilde{\kappa}\xi D_t^b/(1-\xi)} - 1\right]^2 \frac{\tilde{\kappa}\xi D_t^b}{1-\xi},$$

with $\Psi > 0$. For analytical convenience, we express the adjustment cost relative to aggregate bank lending ξD_t^b, which savers take as given.[3]

2.1.2 Borrowers

A generic borrower household $i \in [1-\xi, 1]$ maximizes the same per-period utility as savers (1) but discounts the future at lower rate $\beta_b \in (0, \beta_s)$. The borrower's budget constraint is

$$P_t C_t^b(i) - D_t^b(i) + Q_t H_t^b(i) = W_t^b L_t^b(i) - R_{t-1}^b D_{t-1}^b(i) + Q_t H_{t-1}^b(i) + \Omega_t^b(i) - T_t^b(i),$$

where $D_t^b(i)$ is the amount of borrowing at time t, $T_t^b(i)$ are lump-sum taxes, including those used to obtain an efficient allocation of consumption in the model's steady state, and $\Omega_t^b(i)$ denotes profits from ownership of intermediate goods producing firms.

As common in the literature (e.g Kiyotaki and Moore, 1997), we assume that a collateral constraint limits impatient households' ability to borrow. In particular, their total liabilities cannot exceed a (potentially time-varying) fraction of their current housing wealth

$$D_t^b(i) \leq \Theta_t Q_t H_t^b(i),$$

where $\Theta_t \in [0, 1]$. The term Θ_t represents the maximum loan-to-value (LTV) ratio that borrowers are allowed to undertake. The standard interpretation of such a constraint is that financial intermediaries require borrowers to have a stake in a leveraged investment to prevent moral hazard behaviour. In our policy analysis, we will also entertain the possibility that the macro-prudential authority sets the maximum LTV banks can extend to borrowers. In this sense, the LTV ratio is part of the macro-prudential toolkit that we will study below.

2.2 Banks

A continuum of perfectly competitive banks, indexed by $k \in [0, 1]$, raise funds from savers in the form of deposits and equity (their liabilities), and make loans (their assets) to borrowers. Bank k's balance sheet identity is

$$D_t^b(k) = D_t^s(k) + E_t^s(k). \quad (2)$$

[3] The introduction of this adjustment cost function is a simple way to distinguish bank equity from bank debt (deposits), and captures the idea that deposits are generally more liquid, and thus easier for households to adjust. Little of substance would change in the first-order accurate solution to the model that we examine if we specified bank equity as a state-contingent claim.

In addition, we assume that equity must account for at least a fraction $\tilde{\kappa}_t$ of the total amount of loans banks extend to borrowers

$$E_t^s(k) \geq \tilde{\kappa}_t D_t^b(k). \tag{3}$$

The presence of equity adjustment costs breaks down the irrelevance of the capital structure (the Modigliani-Miller theorem). Savers demand a premium for holding equity, which banks pass on to borrowers in the form of a higher interest rate. From the perspective of the bank, equity is expensive, and thus deposits are the preferential source of funding. In the absence of any constraint, banks would choose to operate with zero equity and leverage would be unbounded. Equation (3) ensures finite leverage for financial intermediaries.

Like for the LTV parameter, we will consider two possible interpretations of $\tilde{\kappa}_t$. The first treats this variable as a shock, relying on the notion that financial institutions target a certain leverage ratio (Adrian and Shin, 2010). According to the second interpretation, while the constraint still plays the role of limiting banks' leverage, it is the macro-prudential authority that sets the capital requirement on financial institutions. In this sense, $\tilde{\kappa}_t$ becomes the second macro-prudential tool for the regulatory authority. Several recent contributions have discussed capital requirements as one of the key instruments to avoid financial crises in the future (e.g. Admati and Hellwig, 2014; Miles et al., 2013). In our analysis, we will focus on the interaction between capital requirements and interest rate setting.

Independently of its interpretation, the capital requirement constraint is always binding in equilibrium, exactly because financial intermediaries seek to minimise their equity requirement. If the capital constraint of all banks were slack, one bank could marginally increase its leverage, charge a lower loan rate, and take the whole market. Therefore, competition drives the banking sector against the constraint.

Banks' profits are

$$\mathcal{P}_t(k) \equiv R_t^b D_t^b(k) - R_t^d D_t^s(k) - R_t^e E_t^s(k) = [R_t^b - (1-\tilde{\kappa}_t)R_t^d - \tilde{\kappa}_t R_t^e] D_t^b(k),$$

where the second equality follows from substituting the balance sheet constraint (2) and the capital requirement (3) at equality. The zero-profit condition implies that the loan rate is a linear combination of the return on equity and the return on deposits

$$R_t^b = \tilde{\kappa}_t R_t^e + (1-\tilde{\kappa}_t) R_t^d,$$

where the time-varying capital requirement represents the weight on the return on equity. A surprise rise in $\tilde{\kappa}_t$, whether due to an exogenous shock or a policy decision, forces banks to delever and raises credit spreads.

2.3 Production

A representative retailer combines intermediate goods according to a technology with constant elasticity of substitution $\varepsilon > 1$

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $Y_t(f)$ represents the intermediate good produced by firm $f \in [0,1]$. Expenditure minimisation implies that the demand for a generic intermediate good is

$$Y_t(f) = \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} Y_t, \tag{4}$$

where $P_t(f)$ is the price of the variety produced by firm f and the aggregate price index is

$$P_t = \left[\int_0^1 P_t(f)^{1-\varepsilon} df\right]^{\frac{1}{1-\varepsilon}}.$$

Intermediate goods producers operate in monopolistic competition, are owned by savers and borrowers according to their shares in the population, and employ labour to produce variety f according to

$$Y_t(f) = A_t L_t(f). \tag{5}$$

Aggregate technology A_t follows a stationary autoregressive process in logs

$$\ln A_t \equiv a_t = \rho_a a_{t-1} + \epsilon_t^a,$$

with $\rho_a \in (0,1)$ and $\epsilon_t^a \sim \mathcal{N}(0, \sigma_a^2)$. To simplify aggregation, we assume $L_t(f)$ is a geometric average of borrower and saver labour, with weights reflecting the shares of the two types

$$L_t(f) \equiv [L_t^b(f)]^{\xi}[L_t^s(f)]^{1-\xi},$$

and the corresponding wage index is

$$W_t \equiv (W_t^b)^{\xi}(W_t^s)^{1-\xi}.$$

Intermediate goods producers set prices on a staggered basis. As customary, we solve their optimisation problem in two steps. First, for given pricing decisions, firms minimise their costs, which implies that the nominal marginal cost M_t is independent of firms characteristics. The second step of the intermediate goods producers problem is to determine their pricing decision. As in Calvo (1983), we assume firms reset their price $\tilde{P}_t(f)$ in each period with a constant probability $1-\lambda$, taking as given the demand for their variety, while the complementary measure of firms λ keeps their price unchanged. The optimal price setting decision for firms that do adjust at time t

solves

$$\max_{\tilde{P}_t(f)} \mathbb{E}_t \left\{ \sum_{v=0}^{\infty} \lambda^i Q_{t,t+v}[(1+\tau^p)\tilde{P}_t(f) - M_{t+v}]Y_{t+v}(f) \right\},$$

subject to (4), where τ^p is a subsidy to make steady state production efficient. Intermediate goods producers are owned by households of both types in proportion to their shares in the population. Therefore, we assume that the discount rate for future profits is

$$Q_{t,t+v} \equiv (Q^b_{t,t+v})^{\xi}(Q^s_{t,t+v})^{1-\xi},$$

where $Q^j_{t,t+v} = ze^{-z(C^j_{t+v}(i)-C^j_t(i))}$ is the stochastic discount factor between period t and $t+v$ of type $j = \{b, s\}$.

2.4 Equilibrium

Because of the assumption of risk-sharing within each group, all households of a given type consume the same amount of goods and housing services and work the same number of hours. Therefore, in what follows, we drop the index i and characterize the equilibrium in terms of type aggregates. Similarly, because all financial intermediaries make identical decisions in terms of interest rate setting, we simply refer to the aggregate balance sheet and abstract from the index k.

For a given specification of monetary and macro-prudential policy, an imperfectly competitive equilibrium for this economy is a sequence of quantities and prices such that households and intermediate goods producers optimise subject to the relevant constraints, final good producers and banks make zero profits, and all markets clear.[4] In particular, for the goods market, total production must equal the sum of consumption of the two types plus the resources spent for portfolio adjustment costs

$$Y_t = \xi C^b_t + (1-\xi)C^s_t + \Gamma_t, \tag{6}$$

where $\Gamma_t \equiv \int_0^{1-\xi} \Gamma_t(i)di$. Similarly, the housing market equilibrium requires

$$H = \xi H^b_t + (1-\xi)H^s_t. \tag{7}$$

Because the aggregate stock of housing plays no further role in the analysis, without loss of generality, in what follows we set $H = 1$. Finally, in the credit market, total bank loans must equal total household borrowing. Thus, the aggregate balance sheet for the financial sector respects

$$\xi D^b_t = (1-\xi)(D^s_t + E^s_t),$$

where per-capita real private debt evolves according to

$$\frac{D^b_t}{P_t} = \frac{R^b_{t-1}}{\Pi_t}\frac{D^b_{t-1}}{P_{t-1}} + C^b_t - Y_t + \frac{Q_t}{P_t}(H^b_t - H^b_{t-1}) + \mathcal{T}^b,$$

[4] Appendix A reports the equilibrium conditions for the private sector and the details of aggregation.

and τ^b is a subsidy that ensures the steady state allocation is efficient.

3 Linear-Quadratic Framework

Our ultimate objective is to study the optimal joint conduct of monetary and macro-prudential policy. To do so, we follow the approach of the optimal monetary policy literature (e.g. Clarida et al., 1999; Woodford, 2003), and derive a linear-quadratic approximation to our model with nominal rigidities and financial frictions. We approximate the model around a zero-inflation steady state in which the collateral constraint binds. An appropriate choice of taxes and subsidies ensures that the steady state allocation is efficient. Appendix C reports the details of the derivations.

3.1 Quadratic Loss Function

In order to derive the welfare-based loss function, we start by averaging the utility function of borrowers and savers weighting each type according to their share in the population. The second-order approximation of the resulting objective is

$$\mathcal{L}_0 \equiv \frac{\sigma+\varphi}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[x_t^2+\lambda_\pi\pi_t^2+\lambda_\kappa\kappa_t^2+\lambda_c(c_t^b-c_t^s)^2+\lambda_h(h_t^b-h_t^s)^2\right], \tag{8}$$

where lower-case variables denote log-deviations from the efficient steady state, $x_t \equiv y_t - y_t^*$ is the efficient output gap, and y_t^* is the efficient level of output. Future losses are discounted at rate $\beta \equiv \xi\beta_b + (1-\xi)\beta_s$, the population-weighted average of borrowers and savers' individual discount factors. The weights on deviations of inflation and capital requirements from target are

$$\lambda_\pi \equiv \frac{\varepsilon}{\gamma} \qquad \text{and} \qquad \lambda_\kappa \equiv \frac{\psi\eta}{\sigma+\varphi},$$

where $\gamma \equiv (1-\beta\lambda)(1-\lambda)(\sigma+\varphi)/\lambda$, while the weights on the consumption and housing gaps are

$$\lambda_c \equiv \frac{\xi(1-\xi)\sigma(1+\sigma+\varphi)}{(1+\varphi)(\sigma+\varphi)} \qquad \text{and} \qquad \lambda_h \equiv \frac{\sigma_h\xi(1-\xi)}{\sigma+\varphi}.$$

The loss function (8) features two sets of terms. The first set includes the efficient output gap and inflation—the standard variables that appear in the welfare-based loss function of a large class of New Keynesian models. Their presence in the loss function reflects the two distortions associated with price rigidities. First, such rigidities open up a "labour wedge", causing the level of output to deviate from its efficient level. Second, staggered price setting implies an inefficient dispersion in prices, which is proportional to the rate of inflation.

The second set of terms in (8), comprising the "consumption gap" $c_t^b - c_t^s$ and the "housing gap" $h_t^b - h_t^s$, arise from the heterogeneity between household types and, in particular, from the fact that one group of households are credit-constrained while the others are not.[5] The reason why the

[5] Consumption for each group is approximated as a percentage of steady-state output ($c_t^j \equiv (C_t^j - C^j)/Y$), while

collateral constraint generates an inefficiency depends on the lack of full insurance. In the absence of the collateral constraint, households could insure each other against variation in their housing and consumption bundles. The collateral constraint limits the amount of borrowing that can take place to carry out this insurance in full. As a result, risk sharing is imperfect. An analogous argument applies to housing. Imperfect risk sharing therefore becomes a source of welfare losses the policymaker wants to take into account when setting policy optimally. In addition, the term $\lambda_\kappa \kappa_t^2$ accounts for the costs associated with the use of capital requirements as a policy tool.[6]

3.2 Linearised Constraints

In this section, we combine the linearised equilibrium relations to obtain a parsimonious set of constraints for the optimal policy problem in our linear-quadratic setting.[7] To simplify the derivations, we assume $\Theta = 1$ (a 100% LTV ratio). We return to the case $\Theta < 1$ in the quantitative analysis. Appendix D provides additional details on the derivations.

On the supply side, as common in the literature, we can rewrite the Phillips curve in terms of the efficient output gap

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^m, \tag{9}$$

where u_t^m is an exogenous cost-push shock.

On the demand side, we write an aggregate demand curve in term of the output gap and the consumption gap

$$x_t - \xi(c_t^b - c_t^s) = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t[x_{t+1} - \xi(c_{t+1}^b - c_{t+1}^s)] + \nu_t^{cgap}, \tag{10}$$

where ν_t^{cgap} is a combination of exogenous shocks defined in the appendix. In a standard representative agent model, the consumption gap is zero, and all agents behave like the savers in our economy. The consumption gap in (10) summarises the impact of debt obligations, house prices and LTV ratios on aggregate demand due to the lack of risk sharing.

In the neighborhood of a steady in which the borrowing constraint binds, debt is a function of the LTV constraint, house prices and the housing gap

$$d_t^b = \theta_t + q_t + (1 - \xi)(h_t^b - h_t^s). \tag{11}$$

We can keep track of its dynamics via the borrowers' budget constraint

$$d_t^b = \frac{1}{\beta_s}(i_{t-1} + \psi \kappa_{t-1} + d_{t-1}^b - \pi_t) + (1 - \xi)[(h_t^b - h_t^s) - (h_{t-1}^b - h_{t-1}^s)] + \frac{1 - \xi}{\eta}(c_t^b - c_t^s). \tag{12}$$

We derive an equation for the housing gap by taking the difference of the housing demand

housing is approximated as percentage of overall housing supply in the economy ($h_t^j \equiv (H_t^j - H^j)/H$).

[6] Obviously, if we treat the leverage ratio as an exogenous—albeit time-varying—constraint, the costs of its fluctuations would become independent of policy, and thus irrelevant for ranking alternative policies in terms of welfare.

[7] Unless otherwise stated, lower-case variables denote log-deviations from steady state. For a generic variable Z_t, with steady state value Z, $z_t \equiv \ln(Z_t/Z)$.

equations between borrowers and savers. The resulting expression is

$$h_t^b - h_t^s = -\frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h \xi \omega}(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\beta_s - \beta_b}{\sigma_h \omega}(q_t - \mathbb{E}_t q_{t+1}) - \frac{\sigma}{\sigma_h \xi}(x_t - \mathbb{E}_t x_{t+1}) + \frac{\sigma}{\sigma_h}(c_t^b - c_t^s) + \frac{\tilde{\mu}}{\sigma_h \omega}\theta_t - \frac{1 - \tilde{\mu}}{\sigma_h \omega}\psi \kappa_t + \nu_t^{hgap}, \quad (13)$$

where ω and ν_t^{hgap} are combinations of fundamental parameters and shocks, respectively, defined in the appendix.

To complete the description of the demand side, we take a population-weighted average of the housing demand equation to obtain an aggregate house price equation that reads as

$$q_t = -(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\sigma \omega}{\omega + \beta}\mathbb{E}_t x_{t+1} + \frac{\xi \tilde{\mu}}{\omega + \beta}\theta_t - \frac{\xi(1 - \tilde{\mu})}{\omega + \beta}\psi \kappa_t + \frac{\beta}{\omega + \beta}\mathbb{E}_t q_{t+1} + \nu_t^h, \quad (14)$$

where ν_t^h is a combination of fundamental shocks defined in the appendix.

The joint optimal monetary and macro-prudential problem consists of minimising (8) subject to the constraints (9)-(14). In the most general case, the policymaker can choose three instruments (the nominal interest rate, i_t, the LTV ratio, θ_t, and the capital requirement, κ_t). In what follows, we will also consider cases in which the macro-prudential authority only sets one instrument (either the LTV ratio or the capital requirement) and treat the other variable as an exogenous shock.

3.3 Optimal Macro-Prudential Policy in the Efficient Equilibrium

In order to highlight the effects of macro-prudential policies, we focus on the efficient equilibrium of the model. With flexible prices ($\lambda \to 0$) and no markup shocks ($u_t^m = 0, \ \forall t$), productivity fully determines output, which becomes exogenous ($y_t = y_t^*$), so that the output gap is always zero ($x_t = 0, \ \forall t$). In addition, the weight on inflation in (8) is zero ($\lambda_\pi = 0$). Hence, the first two terms in the loss function disappear.

The job of the policy authority then becomes to focus on the consumption and housing gap, and—if used as an instrument—minimise the volatility of capital requirements. Suppose that the costs of varying capital requirements are negligible (formally, assume that $\lambda_\kappa \to 0$). Then, the optimal policy plan manages to contemporaneously close both the consumption and housing gap. The key for this result is the ability to generate inflation surprises that stabilise the real value of private debt. Optimal policy does not achieve zero inflation in each state of the world. Instead, the policymaker commits to zero expected rate of inflation

$$\mathbb{E}_t \pi_{t+1} = 0.$$

Because its volatility is costless, ex-post inflation can make the real value of debt state-contingent.

In Appendix E we show that, in this case, private debt solves

$$d_t^b = \frac{1}{\beta_s}\left[\frac{1}{1-\tilde{\mu}} d_{t-1}^b - (\pi_t - \mathbb{E}_{t-1}\pi_t)\right] + \nu_t^b,$$

where ν_t^b is a combination of fundamental shocks that affect debt. The finding that ex-post inflation volatility is necessary for the optimal conduct of monetary and macro-prudential policy is reminiscent of another policy coordination problem, namely the joint setting of fiscal and monetary policy. Under flexible prices, Chari et al. (1991) show that inflation surprises can make the real value of government debt state-contingent and replicate the complete markets allocation. Ex-post inflation carries a similar role in our model, in which the lack of state-contingent transfers between borrowers and savers generates the inefficiency.

This result extends to the case in which changing capital requirements is costly ($\lambda_\kappa > 0$). The policymaker continues to find the use of ex-post inflation surprises optimal so to make private debt state-contingent. In this case, however, contemporaneous stabilisation of the consumption and housing gap is not possible. The policymaker needs to strike a balance between that objective on the one hand, and the volatility of capital requirements on the other.

Appendix E derives two targeting rules that characterise this tradeoff. Under discretion, these rules have a simple representation.[8] The first requires the policymaker to increase capital requirements whenever the housing gap is positive (and vice versa)

$$\lambda_\kappa \kappa_t = \varphi_\kappa \lambda_h (h_t^b - h_t^s), \tag{15}$$

where $\varphi_\kappa > 0$ is a combination of fundamental parameters defined in the appendix.

The second targeting rule calls for moving the consumption and housing gaps in locksteps

$$\lambda_c (c_t^b - c_t^s) = \varphi_h \lambda_h (h_t^b - h_t^s), \tag{16}$$

where $\varphi_h > 0$ is a combination of fundamental parameters also defined in the appendix. This parameter, together with the relative weight on the housing and consumption gaps (λ_h/λ_c), determines the elasticity of the consumption gap to movements of the housing gap induced by changes in capital requirements.

While we can interpret (15) as a targeting rule because of the presence of capital requirements in the loss function, it also directly instructs the policymaker how to set one of its instruments. We can combine the second targeting rule (16) to obtain an instrument rule for the LTV ratio

$$\theta_t = d_t^b - q_t - \frac{(1-\xi)\lambda_c}{\varphi_h \lambda_h}(c_t^b - c_t^s).$$

[8] We present the optimal targeting rules under discretion for simplicity of exposition and for consistency with the quantitative section below. As we explain later, under discretion, the zero lower bound is a more challenging constraint to deal if only monetary policy instruments are available. Hence, in a crisis scenario, the presence of additional macro-prudential tools makes the joint policy problem more interesting to analyse.

Higher house prices and a positive consumption gap require a tightening of the LTV ratio. Against the common wisdom, the instrument rule suggests LTVs should be relaxed in the face of higher debt, for given house prices and consumption gap. While this result may appear counterintuitive, recall that under optimal policy ex-post inflation surprises make debt state-contingent, hence effectively de-emphasising its relevance in this context. Under flexible prices, therefore, monetary policy supports macro-prudential policy, and simplifies its task, via the creation of inflation surprises that minimise the impact of exogenous shocks on private debt.

3.4 Macro-Prudential Policy with Sticky Prices

With sticky prices, the policymaker can no longer resort to ex-post inflation surprises to stabilise the value of private debt. Indeed, going back to the analogy with joint optimal fiscal and monetary policy problems, Siu (2004) demonstrates that the introduction of a small degree of price rigidity leads to little inflation volatility compared to the case of flexible prices. Similarly, in our environment with sticky prices, the policymaker must now deal with the distortions associated with price dispersion, in addition to the lack of insurance that characterises the economy independently of nominal rigidities.

As in the previous section, we report here the optimal targeting rules for monetary and macro-prudential policy under discretion, and refer to Appendix E for the details of the derivation. The first targeting rule is

$$x_t + \gamma\lambda_\pi\pi_t + \frac{\sigma}{\psi}\lambda_\kappa\kappa_t - \mathcal{H}_{\kappa t} = 0, \tag{17}$$

where α_h is a combination of fundamental parameters, and

$$\mathcal{H}_{\kappa t} \equiv \frac{\eta}{1-\xi}\left[\xi\sigma\frac{\lambda_\kappa}{\psi}\kappa_t - \lambda_c(c_t^b - c_t^s) - \xi\alpha_h\lambda_h(h_t^b - h_t^s)\right].$$

The first two terms of (17) correspond to the optimal targeting rule for monetary policy under discretion in the absence of heterogeneity and financial frictions (i.e., in the standard New Keynesian model). The remaining two terms reveal how the presence of imperfect risk sharing affects the conduct of optimal monetary policy. In particular, the gap between the policy instrument and the term $\mathcal{H}_{\kappa t}$ (the "macro-prudential policy gap") summarises the consequences for monetary policy of the impossibility of stabilising contemporaneously the consumption and housing gaps, while minimising the volatility of capital requirements.

The second targeting rule can be written as

$$\frac{\eta}{1-\xi}\zeta_h\lambda_h(h_t^b - h_t^s) - \frac{\lambda_\kappa}{\psi}\kappa_t - \mathcal{H}_{\kappa t} = 0, \tag{18}$$

where ζ_h s a combination of fundamental parameters. The second targeting rule makes clear the link between the macro-prudential policy gap and the housing gap. Differently from the case of flexible prices, the optimal macro-prudential policy responds not only to the housing gap but also to the consumption gap via the term $\mathcal{H}_{\kappa t}$. The intuition is that a combination of capital requirements, the consumption gap, and the housing gap is proportional to output and inflation (compare 17 and 18). Therefore, by responding also to the consumption gap, the policymaker internalises the consequences of monetary policy for macro-prudential policy.

Finally, the last targeting rule has a dynamic connotation

$$\mathcal{H}_{\kappa t} = \frac{\tilde{\mu}\zeta_h}{\sigma_h \omega}\lambda_h (h_t^b - h_t^s) + \beta \mathbb{E}_t \mathcal{H}_{\kappa t+1}. \tag{19}$$

According to this rule, macro-prudential policy needs to take into account current and future expected housing gaps in order to find the appropriate response to current developments.

In sum, the rules (17), (18), and (19) reveal a rich interaction between monetary and macro-prudential policy that operates via the output gap and inflation on the monetary side, and the consumption and housing gap on the macro-prudential side. The next section makes the targeting criteria above operational in the context of a house price boom-bust scenario that mimics the developments of the recent crisis.

4 Quantitative Experiments

In this section, we use our model to study the interaction of monetary and macroprudential policies in a stylized simulation of a house price boom.

To provide a somewhat more realistic dynamic structure of the model, we incorporate a slow-moving borrowing limit in the same way as Guerrieri and Iacoviello (2013) and Justiniano et al. (2015). This modification is intended to generate more persistent movements in debt and its marginal value (that is, the multiplier μ). In principle, it is possible to incorporate a wide range of additional frictions to enhance the dynamic properties of the model (as Guerrieri and Iacoviello, 2013, do, for example). Here, we focus on the slow-moving borrowing limit largely because it does not affect the derivation of the welfare-based loss function while introducing some quantitative relevance.

Specifically, we assume that borrowers face the following borrowing constraint:

$$D_t^b(i) \leq \gamma_d D_{t-1}^b(i) + (1-\gamma_d)\,\Theta_t Q_t H_t^b(i) \tag{20}$$

where $\gamma_d \in [0, 1)$ is a parameter controlling the extent to which the debt limit depends on the household's debt in the previous period. As argued by Guerrieri and Iacoviello (2013), this formulation can be interpreted as capturing the idea that only a fraction of borrowers experience a change to their borrowing limit each period (which may be associated with moving or re-mortgaging). One implication of this formulation is that movements in debt adjust only gradually to changes in the value of the housing stock, which is consistent with the data in Figure 1. The modification to the specification of the borrowing limit affects the Euler equation and housing demand equation of borrowers as shown in Appendix [TBC]. When $\gamma_d = 0$, the model collapses to the version discussion in Section 2.

	Description	Value	Comments/issues/questions
β_s	Saver discount factor	0.995	Guerrieri and Iacoviello (2013)
β_b	Borrower discount factor	0.9922	Guerrieri and Iacoviello (2013)
σ	IES (consumption)	1	Guerrieri and Iacoviello (2013)
φ	Inverse Frisch elasticity	1	Guerrieri and Iacoviello (2013)
γ_d	Debt limit inertia	0.7	Guerrieri and Iacoviello (2013)
γ	Slope of Phillips curve	0.008	Eggertsson and Woodford (2003)
ξ	Fraction of borrowers in economy	0.57	Cloyne et al. (2016)
η	Debt: GDP ratio	1.8	BIS data (1990–2000)
Θ	Debt limit (fraction of house value)	0.7	See text.
ψ	Elasticity of funding cost to capital ratio	0.0125	See text.
σ_h	IES (housing)	5	See text.
ρ_h	Housing demand shock persistence	0.95	See text.

Table 1: Parameter Values

4.1 Parameter Values

The parameter values used for the simulation exercises are shown in Table 1. Some of the parameter values are taken from the careful estimation of a similar (though richer) model on US data by Guerrieri and Iacoviello (2013). We also set the slope of the Phillips curve in line with the assumption in Eggertsson and Woodford (2003).[9] The remaining parameters are set with reference to UK data.

To set ξ we refer to the analysis in Cloyne et al. (2016), who study the behavior of households by tenure type. Their data imply that UK household shares are roughly: 30% homeowners; 40% mortgagors and 30% renters. Since our model does not include renters, we set $\xi = \frac{0.4}{0.3+0.4} \approx 0.57$ so that it represents the relative population shares of mortgagors and homeowners in the data.[10]

We set η with reference to UK household debt to GDP ratios. According to BIS data, this ratio averaged around 60% between 1990 and 2000.[11] Around three quarters of household debt is mortgage debt, which suggests setting $\eta = 0.6 \times 0.75 \times 4 \approx 1.8$ (since η is the ratio of debt to *quarterly* GDP). The average LTV of outstanding UK mortgages was around 50% in 2016, which is substantially lower than the 0.9 estimated by Guerrieri and Iacoviello (2013). Given that estimate and the fact that new mortgages will typically be taken out at higher LTVs, we set $\Theta = 0.7$.

We assume the steady state capital ratio $\widetilde{\kappa}$ is 10%, close to the average reported in Meeks (2017) for UK banks over the period 1990-2008. Given that, the key determinant of the transmission of changes in κ_t through to credit spreads is the parameter $\psi \equiv \Psi\widetilde{\kappa}$. Meeks (2017) estimates that a 50 basis point rise in capital requirements raises mortgages spreads by 20-25 annualised basis points at

[9] Eggertsson and Woodford (2003) assume that the slope of the Phillips curve with respect to the output gap is 0.024. In our model the factor of proportionality between marginal cost and the output gap is $1 + \varphi + \sigma = 3$, so we set γ (the slope of the Phillips curve with respect to marginal cost) to 0.008. This requires a Calvo price adjustment parameter of 0.875, somewhat lower than the estimate of Guerrieri and Iacoviello (2013).

[10] Guerrieri and Iacoviello (2013) estimate that the fraction of labor income accruing to borrowers is around 0.5. Since labor income is allocated in proportion to population share in our model, this suggests a similar value for ξ.

[11] This period precedes the run up in house prices before the financial crisis. We use this period for our calibration as we aim to mimic the pre-crisis house price rise in our simulation.

its peak. Taken together, these assumptions imply a value of Ψ solving $(0.0025/4) = \Psi \times 0.10 \times 0.05$, or equivalently, a value of ψ given by $(0.0025/4)/0.05 = 0.0125$.[12]

Two parameters that are important in determining the response to housing demand shocks are the intertemporal substitution elasticity (σ_h) and the persistence of the shock. We assume a high level of persistence, setting $\rho_h = 0.95$, which is qualitatively consistent with the results in Guerrieri and Iacoviello (2013), though slightly lower than their modal estimate of 0.98. We set $\sigma_h = 5$ which implies that housing demand is relatively insensitive to movements in the real house price. Guerrieri and Iacoviello (2013) assume $\sigma_h = 1$, but also incorporate habit formation in the sub-utility function for housing. The high degree of habit formation that they estimate implies that the short-run elasticity of housing demand to changes in the house price is somewhat lower than unity. By setting $\sigma_h = 5$, we aim to replicate this qualitative behavior without complicating the model and particularly the derivation of the welfare-based loss function.[13]

4.2 A Simulated Housing Boom

Our simulation is designed to generate a prolonged rise in the real price of housing followed by a sharp fall. The simulation is calibrated to deliver a similar increase in real house prices that was observed in pre-crisis UK data (Figure 1). The fall in prices is much more extreme than observed in the UK case because our aim is to generate a sufficiently large downturn that monetary policy may be constrained by the zero bound. In this respect, the simulation resembles perhaps more closely the US experience.

To generate a steady increase in house prices, we assume agents believe that there will be a house price shock u_K^h in period K. For periods $t < K$ this represents a "news" shock. We further assume that agents' beliefs about the size of the shock increase over time, so that $\mathbb{E}_t u_K^h > \mathbb{E}_{t-1} u_K^h$ for $t = 1, \ldots, K-1$. In period K the true value of the housing demand shock is revealed and we assume that $u_K^h < \mathbb{E}_1 u_K^h$, so that housing demand turns out to be much weaker than had been initially expected.

Figure 2 shows the profile of the news shocks and the associated evolution of house price expectations. In particular, panel (a) shows the evolution of $\mathbb{E}_t u_K^h$ for $t = 1, \ldots, K$, while panel (b) shows the associated sequence of expected real house price profiles, measured as percentage deviations from steady state, in the dashed lines. The solid black line in panel (b) shows the actual path of house prices, before and after the realisation of the negative house price shock $u_K < 0$ in period K. To match the slow pre-crisis increase in house prices we set $K = 32$ so that house prices increase for eight years.

[12] Meeks' estimates imply a 50bp increase in the capital requirement reduces the level of GDP by around 0.2% at peak and house prices by around 1%. An alternative calibration strategy aims to match one or both of these responses, and would require a value of ψ higher/lower than assumed here.

[13] Of course, our approach also reduces the long-run elasticity of housing demand to house prices, so that it is less flexible than the introduction of habit formation.

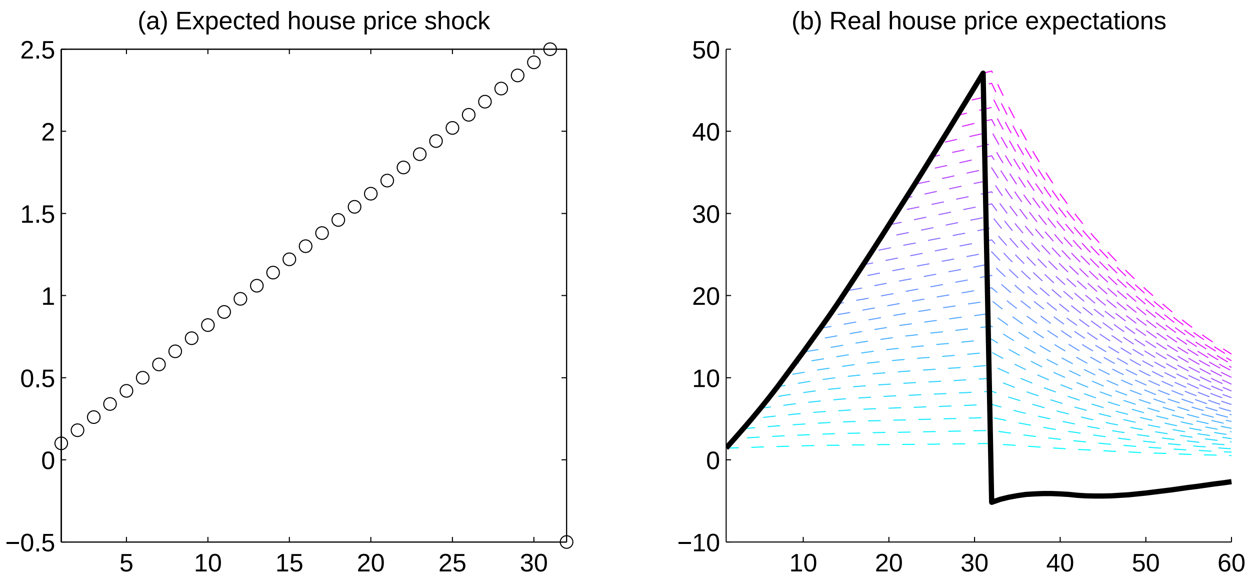

Figure 2: House price news shocks and house price expectations

4.3 Simulation Methodology

We use a piecewise linear solution approach to account for the possibility that (a) the short-term nominal interest rate is constrained by the zero lower bound and/or (b) borrowers' debt limit (20) does not bind. This approach takes account of the possibility that the occasionally binding constraints may apply in future periods, but does not account for the *risk* that future shocks may cause the constraints to bind. This means that our solution approach does not account for the skewness that may be generated in the expected distribution of future outcomes (e.g. of output and inflation) by the possibility of being constrained in future. Our solution methodology is therefore similar to the OccBin toolkit developed by Guerrieri and Iacoviello (2015). Appendix F contains a detailed description of the method we use.

4.4 Flexible Inflation Targeting

Our first experiment considers the case in which a monetary policymaker pursues a 'flexible inflation targeting' mandate, with no macro-prudential policy in place. That is, we assume that the monetary policymaker is tasked with using the short-term nominal interest rate to minimise the loss function

$$\mathcal{L}_0^{FIT} \equiv \frac{\sigma + \varphi}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(x_t^2 + \lambda_\pi \pi_t^2 \right), \tag{21}$$

which corresponds to the welfare-based loss function (8) with the weights on the capital requirement, and on the consumption and housing gaps set to zero. We use $\mathcal{L}_0^{FIT}$ as a simple characterisation of the pre-crisis monetary policy arrangements in which central banks used the short-term nominal interest rate to pursue stabilisation objectives defined in terms of inflation and aggregate real activity.

We assume that the monetary policymaker acts under discretion and is therefore unable to use promises of future policy actions to improve stabilisation outcomes today.[14] In later sections, we will explore the extent to which macro-prudential policies may be able to supplement monetary policy when the latter is constrained by the zero bound. Our assumption that monetary policymakers act under discretion is intended to limit the power of monetary policy at the zero bound. It is well known that commitment policies can be very effective at mitigating the effects of the zero bound in standard New Keynesian models (see, for example, Eggertsson and Woodford, 2003). This setting, therefore, maximises the potential scope for macro-prudential policies to improve outcomes when used alongside monetary policy.

These assumptions imply that, when unconstrained by the zero bound, the monetary policymaker implements a standard flexible inflation-targeting criterion

$$x_t + \gamma \lambda_\pi \pi_t = 0, \tag{22}$$

which is identical to the case of the baseline New Keynesian models (Clarida et al., 1999; Woodford, 2003). Despite the additional richness of our model relative to the canonical New Keynesian model, this 'static' optimality condition is preserved because the policymaker's current decisions have no effect on the ability of future policymakers to set policy optimally. That is, although the model contains endogenous state variables, none of them constrain the ability of future policymakers to stabilise the output gap and inflation by an appropriate choice of the nominal interest rate. However, the monetary policymaker may be constrained by the zero lower bound on the policy rate.

Figure 3 shows the outcome of our simulation when the policymaker pursues flexible inflation targeting. Two variants of the simulation are shown. The solid red lines show the case in which the occasionally binding constraints are ignored. In this case, the multiplier μ on the borrowing constraint is permitted to take negative values as is the nominal policy rate. The dashed blue lines show the case in which the simulation respects the occasionally binding constraints.

The results demonstrate the importance of applying the occasionally binding constraints. When disregarded, the policymaker is able to fully stabilise the output gap and inflation. However, achieving that stabilisation requires quite large fluctuations in the nominal interest rate. Indeed, the collapse in housing demand generates a long period in which the nominal interest rate is negative. Moreover, during the period of increasing house prices, nominal interest rates rise by around two percentage points. This reaction is required in order to stabilise aggregate demand, which is supported by increased consumption demand by borrowers, given that the multiplier on the borrowing constraint enters negative territory.

[14] Formally, we solve for a Markovian policy. In each period, the policymaker acts as a Stackelberg leader with respect to private agents and future policymakers. The current policymaker takes the decision rules of future policymakers as given. In equilibrium, the decisions of the policymaker in the current period satisfy the decision rule followed by future policymakers. See Appendix F for technical details.

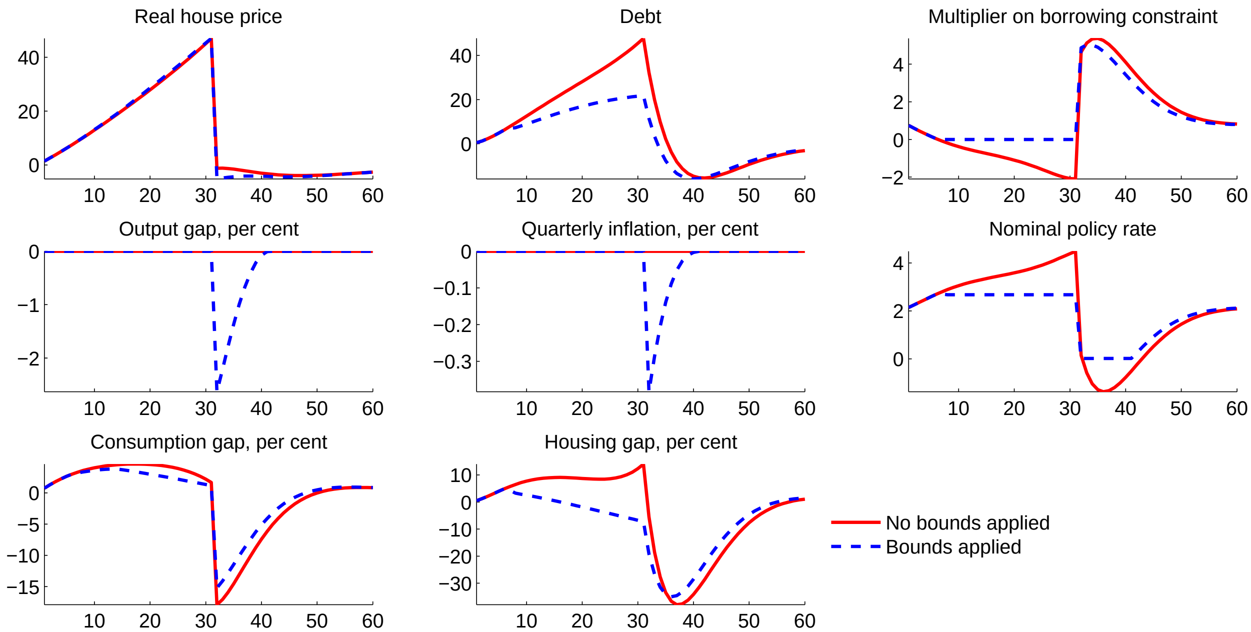

Figure 3: 'Flexible inflation targeting' loss function

When the occasionally binding constraints are imposed, it is no longer possible for the monetary policymaker to stabilise the output gap and inflation when house prices fall. There is a recession and a sharp decline in inflation while the nominal interest rate is constrained by the zero lower bound. The occasionally binding constraints are also important during the house price boom. The expectation of strong real house prices causes the borrowing constraint to go slack (so that the multiplier μ equals zero for a number of periods). Relative to the case in which the constraints are not applied (solid red lines), there is a smaller increase in debt and the consumption gap is also smaller. The more moderate spending behaviour of borrowers puts less pressure on aggregate demand so that (before the house price collapse) aggregate demand and inflation are stabilised with a relatively modest increase in the nominal interest rate.

The broad contours of the housing boom in our stylised simulation match some of the qualitative features of the Great Moderation period: house prices rose strongly (and debt increased substantially, albeit by less than house prices), but the output gap and inflation were well stabilised with relatively low nominal interest rates.

4.5 Flexible Inflation Targeting and Macro-Prudential Policy

While the flexible inflation targeting approach is relatively successful in stabilising the output gap and inflation during the house price boom, Figure 3 reveals marked movements in the other components of the welfare-based loss function, namely the consumption gap and housing gap. As house prices rise, a positive consumption gap is created by an increase in the consumption of borrowers (and a fall in the consumption of savers). During the period in which the borrowing constraint is slack, the housing gap becomes negative. When the borrowing constraint is slack,

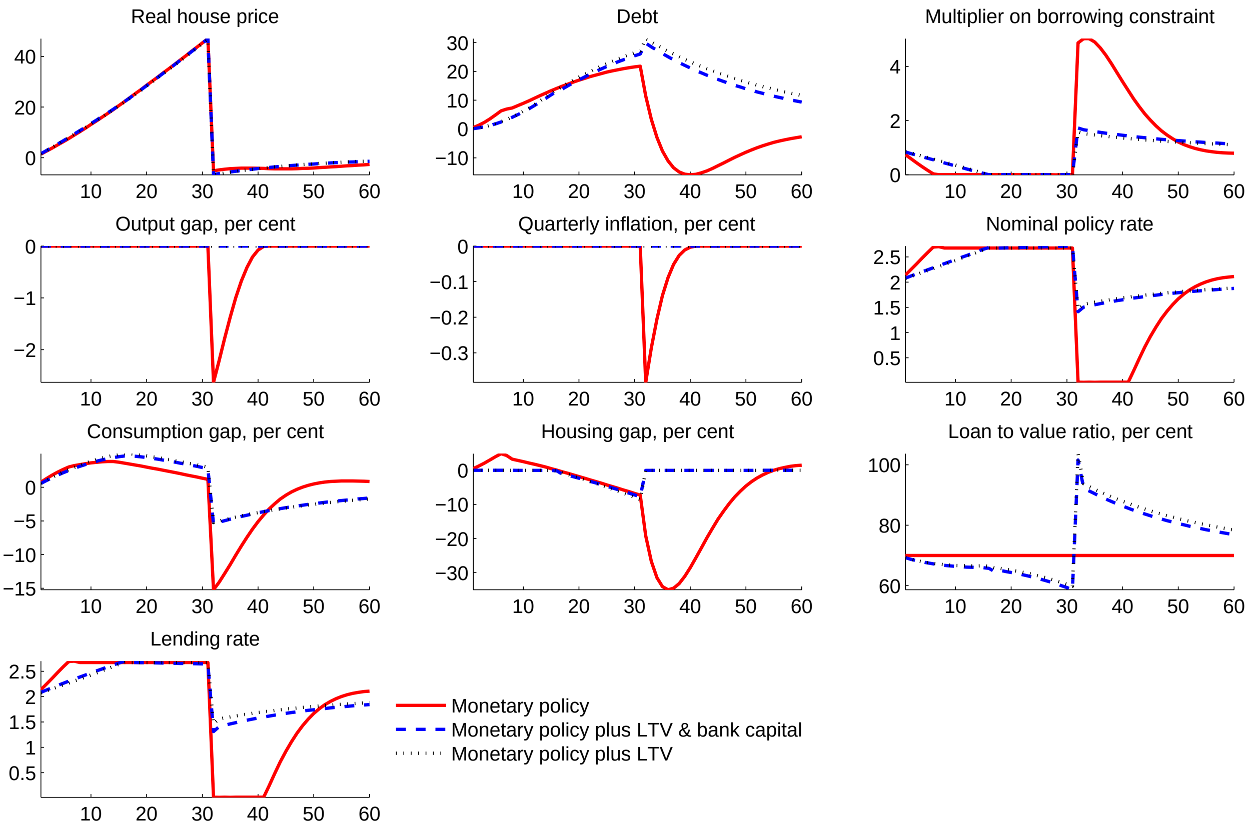

Figure 4: Monetary policy with and without macroprudential policy

there is no additional collateral value in housing so that borrowers are prepared to substitute housing services for non-durable consumption.

These observations suggest that welfare could be improved by using macro-prudential policies. To examine this case, we now assume that monetary policy continues to implement the flexible inflation targeting rule (22) (unless constrained by the zero lower bound) but also that there is a macro-prudential policymaker. The macro-prudential policymaker uses the loan to value ratio (θ) and the bank capital ratio (κ) to minimise the 'macro-prudential' component of the welfare-based loss function

$$\mathcal{L}_0^{MP} \equiv \frac{\sigma+\varphi}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_\kappa \kappa_t^2 + \lambda_c (c_t^b - c_t^s)^2 + \lambda_h (h_t^b - h_t^s)^2 \right], \tag{23}$$

We assume that the macro-prudential policymaker sets the instrument(s) under discretion, mirroring our assumption for monetary policy. Figure 4 compares outcomes when macro-prudential policy is in operation with those in which just monetary policy is in use.[15]

Consider first the case in which the loan to value (LTV) ratio, θ, is used alongside monetary policy. Using the LTV ratio as an instrument enables complete stabilisation of the output gap and inflation (dotted black lines). There is no recession when the real house price falls because

[15] All simulations in this section impose the relevant occasionally binding constraints.

macro-prudential policy is aggressively loosened—the LTV ratio is increased to more than 100%. The loosening of borrowing conditions relaxes the borrowing constraint, so that the multiplier μ rises by much less than in the case in which only monetary policy is used (solid red lines). This implies that a much smaller cut in the nominal policy rate is required to stabilise the output gap and inflation, so that monetary policy is not constrained by the zero lower bound. The loosening of macro-prudential policy when house prices fall results in a much slower fall in aggregate debt (less deleveraging), which helps to avoid large movements in the consumption and housing gaps. Since borrowers are not required to reduce their debt levels as rapidly as would be the case with a fixed LTV, their consumption does not fall as drastically.

The use of the LTV as a macro-prudential policy instrument also has implications for macroeconomic dynamics during the housing boom. In the initial phase, a tightening of macro-prudential policy limits the increase in debt and succeeds in stabilising the housing gap. However, once house prices further accelerate and the borrowing constraint becomes slack ($\mu = 0$), a housing gap opens up. At this point, the LTV instrument is no longer able to affect equilibrium allocations, since θ affects outcomes by moving the borrowing constraint. The LTV continues to fall during the later phase of the housing boom because we assume that it is determined by the condition that the borrowing constraint *just* binds.[16] But during this later phase of the boom, tighter macro-prudential policy is unable to stabilise the housing gap. Consistent with these observations, once the borrowing constraint has become slack, the pace of debt accumulation increases, generating a larger consumption gap (relative to the case in which just monetary policy is used).

Turning to the case in which the bank capital ratio κ is used alongside the LTV ratio (blue dashed lines), we observe very little difference from the case in which just the LTV ratio is used for macro-prudential purposes. The main reason for this result is that the effect of changes in the capital ratio on spreads (ψ) is very small. This means that very large changes in capital requirements are required to have a substantial influence on spreads. But large changes in κ also incur a direct cost in the 'macro-prudential' loss function $\mathcal{L}_0^{MP}$. The net effect is that the capital ratio is adjusted relatively little, with a correspondingly small marginal effect on outcomes. However, the qualitative direction in which the capital ratio is adjusted mirrors that of the LTV. During the housing boom, more stringent capital requirements increase the spread of the borrowing rate over the saving rate. Conversely, after the house price fall, laxer capital requirements reduce the spread.

4.6 Introducing Macro-Prudential Policy after a House Price Crash

The simulation in Section 4.5 considered a case in which macro-prudential policy instruments were available from the outset. However, many countries introduced macro-prudential policy frameworks as a response to the financial crisis. The instruments were simply not part of the policy toolkit during the boom periods.

In this section, we examine whether introducing macro-prudential policies in the aftermath of a

[16] We can assume that for the LTV instrument to be operative there is a slightly positive lower bound on μ, which determines the LTV that must be chosen to implement that condition.

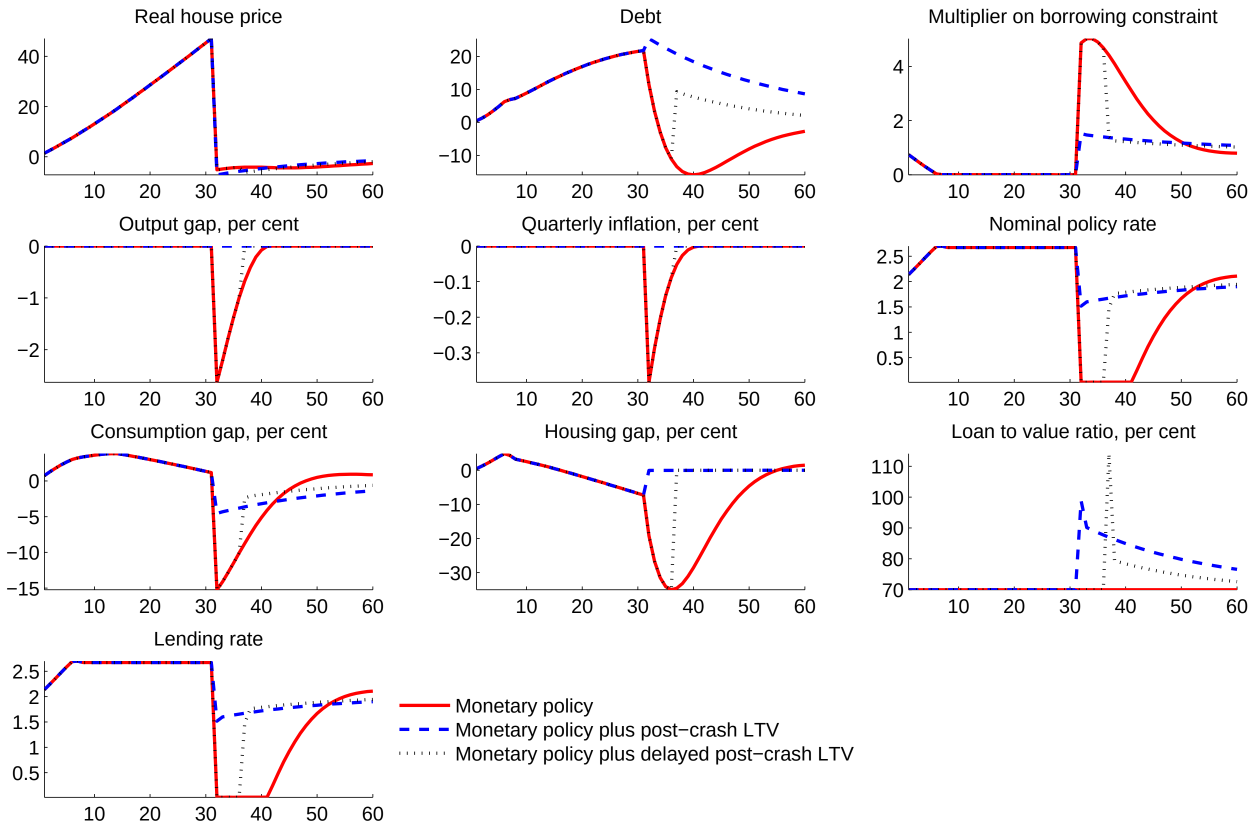

Figure 5: Monetary policy with macroprudential policy after the house price fall

sharp house price fall would constrain their ability to help stabilise the economy. To investigate this possibility, we suppose that only monetary policy is in operation until period $N \geq K$. Until period N agents place a zero probability on the introduction of a macro-prudential policy regime. However, in period N the macro-prudential policymaker is in fact created and instructed to minimise $\mathcal{L}_t^{MP}$ (the 'macro-prudential' loss function).[17]

We consider two alternative experiments. In the first case, we set $N = K = 32$ so that the macro-prudential policy regime is implemented in the same period as the sharp fall in house prices. In the second case, we set $N = 37$, so that implementation of the the macro-prudential policy regime occurs during the recession following the fall in house prices.

The results in Figure 5 show that the introduction of a macro-prudential policy regime leads to an immediate improvement in outcomes regardless of whether it is implemented in the period during which house prices collapse (dashed blue lines) or during the recession (dotted black lines). In both cases, a sharp loosening in macro-prudential policy is capable of eliminating the zero bound constraint and closing the housing gap.

Two aspects of the results in Figure 5 are worth noting. First, very extreme increases in the

[17] This approach implies that our simulation falls foul of the variant of the Lucas (1976) critique set out by Cooley et al. (1984). To address their critique we would need to explicitly incorporate agents' *ex-ante* beliefs about the probability of a macro-prudential regime being implemented in period N.

LTV ratio are required to 're-lever' borrowers. When the introduction of the macro-prudential policy regime is delayed, the LTV is increased to more than 100% of the value of the borrower's collateral, which may be regarded as implausible in the context of real-world collateral constraints. The second observation is that the trajectories of the key variables (particularly the 'goal' variables in the monetary and macro-prudential loss functions) are very similar, regardless of the date on which the regime is introduced. However, the trajectories of debt and the LTV instrument are somewhat different. This highlights the fact that the consumption of borrowers is constrained by a composite (quasi-difference) debt servicing variable. As a result, the level of debt is not a sufficient statistic for assessing the extent to which the borrowing constraint is impacting borrowers' decisions.[18]

5 Conclusion

We use a simple model to examine the interaction of monetary and macro-prudential policies. Our model is rich enough to generate meaningful policy tradeoffs, but sufficiently simple to deliver tractable expressions for welfare and analytical results under some parameterisations. We derive a welfare-based loss function as a quadratic approximation to a weighted average of the utilities of borrowers and savers.

We use the model to study how monetary and macro-prudential policies (LTV ratios and capital requirements) should optimally respond to shocks. To build intuition, we derive some analytical results under restrictive assumptions on parameters and nature of the constraints. In this simplified setting, we demonstrate that macro-prudential policy generally faces a trade-off in stabilising the distribution of consumption and the distribution of housing services, even when prices are flexible and both macro-prudential tools are available to use. We also show that monetary policy alone has relatively little control over these distributions, particularly the distribution of housing between borrowers and savers. In other words, imperfect risk sharing is a real phenomenon whose consequences could be addressed by macro-prudential policies. Nevertheless, these policies also imply costs that must be accounted for in deploying them. This tradeoff prevents complete macro-prudential stabilisation given the tools we study even under flexible prices.

We use the model to explore a simulation of a prolonged boom followed by a sharp fall in house prices. When there is a single monetary policymaker pursuing a 'flexible inflation targeting' mandate (minimising a loss function that includes only the output gap and inflation), the house price fall causes a recession because monetary policy is constrained by the zero bound. During the housing boom, increases in the policy rate required to stabilise the output gap and inflation are moderated because the borrowing constraint becomes slack. Although the output gap and inflation are fully stabilised during the boom, welfare losses are incurred because monetary policy is unable

[18] A better summary statistic would be the multiplier on the borrowing constraint μ which also follows a very similar trajectory once the macro-prudential policy regime is introduced. As long as the borrowing constraint binds, macro-prudential policy can be thought of as directly choosing μ, with the borrowing constraint determining the value of θ required to support it.

to stabilise the consumption and housing gaps. Allowing macro-prudential policies to focus on stabilising the consumption and housing gaps improves welfare substantially. Indeed, the existence of macro-prudential policy implies that there is no recession after the house price fall: monetary policy is able to stabilise the output gap and inflation without hitting the zero bound.

Our results bear important consequences for the current economic environment. As economic conditions show promising improvements in many countries, and central banks prepare to raise interest rates away from the effective lower bound, macro-prudential decisions may have non-negligible effects on the speed of the recovery. In particular, a significant tightening of either maximum LTV ratios or capital requirements (or both), while justified on other grounds, may contribute to delay the recovery, if not generating an outright recession. Indeed, it would seem prudent, as our initial quote of Carney (2014) suggests, to use all policy tools in concert.

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Appendix

A Private Sector Optimality Conditions and Aggregation

This appendix reports the optimality conditions of the private sector (savers, borrowers, and intermediate goods producers) and the details of the aggregation.

A.1 Savers

Starting with savers, the first order condition for deposits is

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^d_t}{\Pi_{t+1}}\right]=1,$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. The corresponding condition for bank equity is

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^e_t}{\Pi_{t+1}}\right]=1+\Psi\left[\frac{E^s_t(i)}{\tilde{\kappa}\xi D^b_t/(1-\xi)}-1\right].$$

Combining the two Euler equation, we can obtain the no-arbitrage condition between equity and deposits

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^e_t-R^d_t}{\Pi_{t+1}}\right]=\Psi\left[\frac{E^s_t(i)}{\tilde{\kappa}\xi D^b_t/(1-\xi)}-1\right].$$

After rearranging, the first order condition for housing services can be written as

$$(1+\tau^h)\frac{Q_t}{P_t}=\frac{\chi^s_H H^s_t(i)^{-\sigma_h}}{e^{-zC^s_t(i)}}+\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{Q_{t+1}}{P_{t+1}}\right].$$

The labour supply condition is

$$W^s_t=\frac{\chi^s_L L^s_t(i)^{\varphi}}{e^{-zC^s_t(i)}}.$$

The budget constraint at equality completes the list of first order conditions for savers.

A.2 Borrowers

Moving on to borrowers, we attach a Lagrange multiplier normalised by the real marginal utility of consumption ($\tilde{\mu}_t(i)ze^{-zC^b_t(i)}/P_t$) to the collateral constraint. The first order condition for borrowed funds is

$$\mathbb{E}_t\left[\beta_b e^{-z(C^b_{t+1}(i)-C^b_t(i))}\frac{R^b_t}{\Pi_{t+1}}\right]=1-\tilde{\mu}_t(i).$$

The first order condition for housing demand is

$$(1-\Theta\tilde{\mu}_t(i))\frac{Q_t}{P_t}=\frac{\chi^b_H H^b_t(i)^{-\sigma_h}}{e^{-zC^b_t(i)}}+\mathbb{E}_t\left[\beta_b e^{-z(C^b_{t+1}(i)-C^b_t(i))}\frac{Q_{t+1}}{P_{t+1}}\right].$$

The labour supply condition is

$$W_t^b = \frac{\chi_L^b L_t^b(i)^{\varphi}}{e^{-zC_t^b(i)}}.$$

The equilibrium conditions for borrower households include the complementary slackness condition

$$\tilde{\mu}_t(i)[D_t^b - \Theta_t Q_t H_t^b(i)] = 0.$$

The budget constraint at equality completes the list of first order conditions for borrowers.

A.3 Firms

The text reports the optimality condition for banks, which is the result of perfect competition in the financial sector.

To derive the expression for the marginal cost, we solve the dual problem

$$\min_{L_t^b, L_t^s} \frac{W_t^b}{P_t} L_t^b(f) + \frac{W_t^s}{P_t} L_t^s(f),$$

subject to the technological constraint given by the production function. Let $M_t(f)$ be the multiplier on the constraint (the real marginal cost). The first order conditions for the two types of labour are

$$\begin{aligned} \frac{W_t^b}{P_t} &= \xi M_t(f) A_t L_t^b(f))^{\xi-1} L_t^s(f)^{1-\xi} &&= \xi M_t(f) \frac{Y_t(f)}{L_t^b(f)} \\ \frac{W_t^s}{P_t} &= (1-\xi) M_t(f) A_t L_t^b(f))^{\xi} L_t^s(f)^{-\xi} &&= (1-\xi) M_t(f) \frac{Y_t(f)}{L_t^s(f)}. \end{aligned}$$

Taking the ratio between the two first order conditions above shows that at the optimum all firms choose the same proportion of labor of the two types. As a consequence, the marginal cost is independent of firm-specific characteristics ($M_t(f) = M_t$). Furthermore, if we take a geometric average of the two first order conditions above, with weights ξ and $1-\xi$, respectively, we obtain the expression for the marginal cost

$$M_t = \frac{W_t/P_t}{\xi^{\xi}(1-\xi)^{1-\xi} A_t},$$

where the expression for the aggregate wage index is reported in the text.

Intermediate goods producers set prices on a staggered basis. Their optimality condition can be summarised by a non-linear Phillips curve

$$\frac{X_{1t}}{X_{2t}} = \left(\frac{1-\lambda \Pi_t^{\varepsilon-1}}{1-\lambda}\right)^{\frac{1}{-\varepsilon}},$$

where X_{1t} represents the present discounted value of real costs

$$X_{1t} = \frac{\varepsilon}{\varepsilon - 1} z e^{-zC_{t+1}} Y_t M_t + \beta\lambda \mathbb{E}_t(\Pi_t^{\varepsilon} X_{1t+1}),$$

and X_{2t} represents the present discounted value of real revenues

$$X_{2t} = (1 + \tau^p) z e^{-zC_{t+1}} Y_t + \beta\lambda \mathbb{E}_t(\Pi_t^{\varepsilon - 1} X_{2t+1}).$$

A.4 Aggregation

To aggregate within types, we simply integrate over the measure of households in each group. Consumption of savers and borrowers is

$$\int_0^{1-\xi} C_t^s(i) di = (1-\xi) C_t^s \qquad \text{and} \qquad \int_{1-\xi}^{1} C_t^b(i) di = \xi C_t^b,$$

while housing demand is

$$\int_0^{1-\xi} H_t^s(i) di = (1-\xi) H_t^s \qquad \text{and} \qquad \int_{1-\xi}^{1} H_t^b(i) di = \xi H_t^b.$$

In the credit market, total bank loans must equal total household borrowing

$$\int_0^{1} D_t^b(k) dk = \int_{1-\xi}^{\xi} D_t^b(i) di = \xi D_t^b.$$

Similarly, for deposits and equity holdings we have

$$\int_0^{1} D_t^s(k) dk = \int_0^{1-\xi} D_t^s(i) di = (1-\xi) D_t^s \qquad \text{and} \qquad \int_0^{1} E_t^s(k) dk = \int_0^{1-\xi} E_t^s(i) di = (1-\xi) E_t^s.$$

Using these expressions, we obtain the aggregate balance sheet for the financial sector and the economy-wide capital constraint reported in the text.

Labour market clearing requires

$$\int_0^{1} L_t^s(f) df = \int_0^{1-\xi} L_t^s(i) di = (1-\xi) L_t^s \qquad \text{and} \qquad \int_0^{1} L_t^b(f) df = \int_{1-\xi}^{1} L_t^b(i) di = \xi L_t^b.$$

Aggregating production across firms yields

$$\int_0^{1} Y_t(f) df = \int_0^{1} A_t L_t^b(f)^{\xi} L_t^s(f)^{1-\xi} df. \tag{24}$$

As discussed in the previous section, the ratio of hours worked of different types is independent of firm-specific characteristics. Therefore, using the labour market equilibrium conditions, we can rewrite the right-hand side of the previous expression as

$$\int_0^1 A_t L_t^b(f)^\xi L_t^s(f)^{1-\xi} df = A_t(\xi L_t^b)^\xi((1-\xi)L_t^s)^{1-\xi} = \xi^\xi(1-\xi)^{1-\xi} A_t L_t,$$

where aggregate labour is

$$L_t \equiv (L_t^b)^\xi (L_t^s)^{1-\xi}. \tag{25}$$

Using the demand for firm f's product, the left-hand side of (24) can also be rewritten in terms of aggregate variables only as

$$\int_0^1 Y_t(f)df = \Delta_t Y_t,$$

where Δ_t is an index of price dispersion, defined as

$$\Delta_t \equiv \int_0^1 \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} df.$$

Given the definition of the price index and the assumption of staggered price setting, the index of price dispersion evolves according to

$$\Delta_t = \lambda \Delta_{t-1} \Pi_t^\varepsilon + (1-\lambda)\left(\frac{1-\lambda\Pi_t^{\varepsilon-1}}{1-\lambda}\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

Therefore, in the aggregate, production is described by

$$\Delta_t Y_t = \xi^\xi(1-\xi)^{1-\xi} A_t L_t.$$

The last step of the aggregation is the derivation of the law of motion of debt. To obtain this equation, we start from the flow budget constraint of a generic borrower

$$P_t C_t^b(i) - D_t^b(i) + Q_t H_t^b(i) = W_t^b L_t^b(i) - R_{t-1}^b D_{t-1}^b(i) + Q_t H_{t-1}^b(i) + \Omega_t^b(i) - T_t^b(i).$$

We assume that each household $i \in [0,1]$ receives an equal share of aggregate value added

$$\Omega_t^j(i) = P_t Y_t - W_t L_t,$$

for $j = \{b, s\}$. From the first order conditions of intermediate goods producers we have

$$W_t^b L_t^b(f) = \xi M_t Y_t(f) \qquad \text{and} \qquad W_t^s L_t^s(f) = (1-\xi) M_t Y_t(f).$$

Integrating over firms, we obtain

$$W_t^b L_t^b = W_t^s L_t^s = W_t L_t = M_t \Delta_t Y_t,$$

where we have used the labour market equilibrium conditions, the definition of the wage and labour indexes, and the definition of the price dispersion index.

Aggregating the borrowers' individual budget constraints, we can then write

$$C_t^b - \frac{D_t^b}{P_t} + \frac{Q_t}{P_t} H_t^b = Y_t - \frac{R_{t-1}^b}{\Pi_t} \frac{D_{t-1}^b}{P_{t-1}} + \frac{Q_t}{P_t} H_{t-1}^b - \mathcal{T}^b, \tag{26}$$

where $\mathcal{T}^b$ is a steady state net tax/subsidy, which includes the borrowers' contribution to the firms' subsidy that make steady state output efficient, and the subsidy borrowers receive to obtain an efficient allocation. We can rewrite the last expression to capture the law of motion of debt

$$\frac{D_t^b}{P_t} = \frac{R_{t-1}^b}{\Pi_t} \frac{D_{t-1}^b}{P_{t-1}} + C_t^b - Y_t + \frac{Q_t}{P_t}(H_t^b - H_{t-1}^b) + \mathcal{T}^b.$$

B Efficient Steady State

This section first establishes the conditions under which a zero inflation ($\Pi = 1$) steady state is efficient, and then discusses how we can obtain efficiency of the steady state allocation in the decentralised equilibrium.[19]

Consider a social planner who maximises a weighted average of borrowers and savers' per-period welfare

$$\mathbb{U} \equiv \tilde{\xi} U(C^b, H^b, L^b) + (1 - \tilde{\xi}) U(C^s, H^s, L^s), \tag{27}$$

for some Pareto weights $\tilde{\xi} \in [0, 1]$, where $U(C^j, H^j, L^j)$ is the per-period utility function of type $j = \{b, s\}$. The social planner chooses allocations subject to the constraints imposed by the aggregate production function and the market clearing conditions for goods, housing, and labour. Importantly, the planner is not subject to the borrowing constraint.

In steady state, there is no price dispersion ($\Delta = 1$). We can further normalise steady state productivity A to one and combine the production function with the goods and labour market constraints to yield

$$(L^b)^\xi (L^s)^{1-\xi} = \xi C^b + (1 - \xi) C^s.$$

Let μ_1 be the Lagrange multiplier on this constraint and μ_2 be the multiplier on the housing resource constraint. The first-order conditions for an efficient steady state are

$$\begin{aligned}
\tilde{\xi} U'_{C^b} &= \mu_1 \xi \\
(1 - \tilde{\xi}) U'_{C^s} &= \mu_1 (1 - \xi) \\
\tilde{\xi} U'_{H^b} &= \mu_2 \xi \\
(1 - \tilde{\xi}) U'_{H^s} &= \mu_2 (1 - \xi) \\
\tilde{\xi} U'_{L^b} &= \mu_1 \xi Y / L^b \\
(1 - \tilde{\xi}) U'_{L^s} &= \mu_1 (1 - \xi) Y / L^s
\end{aligned}$$

[19]Without loss of generality, we normalize the price level to one so that all variables can be thought of as expressed in real terms.

If the Pareto weights coincide with the population weights ($\tilde{\xi} = \xi$), the marginal utility of consumption and housing are equal across types

$$U'_{C^b} = U'_{C^s} = \mu_1 \qquad \text{and} \qquad U'_{H^b} = U'_{H^s} = \mu_2,$$

and so are their levels. In addition, if the disutility of labour has a constant elasticity of substitution, as we assumed, hours supplied by borrowers and savers are proportional to each other depending on the disutility parameters χ^s and χ^b.

For a given type of household, we also obtain

$$\frac{U'_{C^j}}{U'_{H^j}} = \frac{\mu_1}{\mu_2}.$$

The ratio of the marginal utilities of consumption and housing for the two types are the same. The efficient steady state also implies the usual optimality conditions that equates the marginal rate of substitution between consumption and leisure to the marginal rate of transformation between labour and output

$$\frac{U'_{L^j}}{U'_{C^j}} = \frac{Y}{L^j}.$$

Assuming the subsidy τ^p is set as to remove the distortions from monopolistic competition in steady state ($M = 1$), the labour market equilibrium implies

$$\left[\frac{\chi_L^b (L^b)^{\varphi}}{z \exp(-zC^b)}\right]^{\xi} \left[\frac{\chi_L^s (L^s)^{\varphi}}{z \exp(-zC^s)}\right]^{1-\xi} = \frac{Y}{L}.$$

Using the goods and labour market clearing conditions, and replacing output with labour from the production function, equilibrium hours solve

$$L^{\varphi} \exp(zL) = \frac{z}{\left(\chi_L^b\right)^{\xi} \left(\chi_L^s\right)^{1-\xi}}.$$

We can choose the labour supply disutility parameters χ_L^j to deliver a desired target for hours worked by each group (e.g. 2/3 of the households' time endowments). Given this result, the production function pins down the equilibrium level of output. Therefore, importantly, the steady state efficient level of output and hours is independent of the distribution of wealth/debt across household types.

The next step is to find conditions under which the steady state allocation of the decentralised economy is efficient. In particular, we seek the taxes that achieve this objective. In the steady state of the decentralised economy, the savers' discount rate pins down the real rate of interest

$$R^d = \frac{1}{\beta_s}.$$

Since the ratio between equity and deposits is at its desired level, the spread between the return on equity and the return on deposits is zero, and so is the spread between loan and deposit rates

$$R^b = R^e = R^d.$$

In what follows, we drop the superscripts from returns and simply call the steady state gross real interest rate R. From the Euler equation for borrowers, we can obtain the value of the Lagrange multiplier on the collateral constraint

$$\tilde{\mu} = 1 - \beta_b R,$$

which is positive as long as our initial assumption $\beta_b < 1/R = \beta_s$ is satisfied (that is, borrowers are relatively impatient).[20] With a positive multiplier, the constraint binds, and so equilibrium debt is

$$D^b = \Theta Q H^b.$$

Finally, we turn to the housing block. Starting from the law of motion of debt in steady state, we can write

$$C^b = Y - \mathcal{T}^b - (R^b - 1)D^b.$$

In an efficient steady state, the level of consumption must be equal ($C^b = C^s$). Therefore, from the resource constraint, we have that $C^b = C^s = Y = C$. Substituting into the previous condition yields

$$\tau^b = -\frac{1-\beta_s}{\beta_s}\eta,$$

where $\eta \equiv D^b/Y$ is the ratio of debt to GDP and $\tau^b \equiv \mathcal{T}^b/Y$ is subsidy to borrowers (net of their contribution to the production subsidy) that equalises consumption across types.

The last element that we need to determine is the housing tax τ^h. In steady state, the housing demand equation for borrowers is

$$(1 - \Theta\tilde{\mu} - \beta_b)Q = \frac{\chi_H^b (H^b)^{-\sigma_h}}{e^{-zC}},$$

while for savers we have

$$(1 + \tau^h - \beta_s)Q = \frac{\chi_H^s (H^s)^{-\sigma_h}}{e^{-zC}},$$

where we have used the equality of consumption across types. For the steady state housing allocation to be efficient, we must have that the numerator of the right-hand side of the last two expressions (the marginal utility of housing) is equal across types.[21] Therefore, the steady state housing tax must be

$$\tau^h = (\beta_s - \beta_b)\left(1 - \frac{\Theta}{\beta_s}\right),$$

[20] Alternatively, we could write the value of the Lagrange multiplier on the borrowing constraint as $\tilde{\mu} = (\beta_s - \beta_b)/\beta_s > 0$, as long as $\beta_s > \beta_b$, which again corresponds to the initial assumption on the individual discount factors.

[21] In addition, if the housing preference parameters are the same across households ($\chi^b = \chi^s = \chi$), then also the actual level of housing services consumed is the same ($H^b = H^s = H$).

where we used the expression for the steady state Lagrange multiplier $\tilde{\mu} = 1 - \beta_b/\beta_s$, which we obtained by combining the Euler equations for credit and deposits of the two types. Note that, the steady state tax on housing is zero if either $\Theta = \beta_s$ or in the limit $\beta_b \to \beta_s$.

B.1 Macro-Prudential Policy in the Efficient Equilibrium

This section shows that, in a flexible-price efficient equilibrium, macro-prudential policy carries distributional consequences but has no impact on the level of aggregate activity.

As derived in Appendix A, labour supply for type j's satisfies

$$W_t^j = \frac{\chi_L^j (L_t^j)^\varphi}{z \exp(-zC_t^j)}.$$

Weighting the labour supply of each type by their respective shares, using the definition of the wage index, and equating with labour demand gives

$$\left[\frac{\chi_L^b (L_t^b)^\varphi}{z \exp(-zC_t^b)}\right]^\xi \left[\frac{\chi_L^s (L_t^s)^\varphi}{z \exp(-zC_t^s)}\right]^{1-\xi} = \frac{Y_t}{L_t}$$

Using the definition of the labor aggregator and the resource constraint, we can simplify the previous expression to

$$\frac{\left(\chi_L^b\right)^\xi \left(\chi_L^s\right)^{1-\xi} L_t^{1+\varphi}}{z \exp[-z(Y_t - \Gamma_t)]} = Y_t,$$

where Γ_t is the portfolio adjustment cost term. In a flexible-price efficient equilibrium, the aggregate production function is simply $Y_t = A_t L_t$. Therefore, we can express the last condition in terms of the efficient level of output Y_t^* as

$$\frac{\left(\chi_L^b\right)^\xi \left(\chi_L^s\right)^{1-\xi}}{z \exp(-zY_t^*)} \left(\frac{Y_t^*}{A_t}\right)^{1+\varphi} = Y_t^*. \tag{28}$$

In principle, portfolio adjustment costs associated with savers' debt-equity choice do affect output under flexible prices, but this effect is second order because Γ_t is quadratic. As a result, to a first order approximation, the efficient level of output only depends on technology and preference parameters.

In spite of no first-order effects on aggregate supply, macro-prudential measures retain distributional consequences even in an efficient equilibrium. The macro-prudential authority will not be indifferent between different levels of LTV ratios or capital requirements. We return to this point in the next section after deriving a linear-quadratic approximation of the model that allows us to study the optimal joint conduct of monetary and macro-prudential policy. The flexible-price efficient equilibrium will be a useful starting point for our analysis.

C Derivation of the Loss Function

The welfare-based loss function is:

$$\mathcal{L}_0 \equiv -\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_y \left(y_t - y_t^* \right)^2 + \lambda_\pi \pi_t^2 + \lambda_c (c_t^b - c_t^s)^2 + \lambda_h (h_t^b - h_t^s)^2 \right],$$

where lower-case variables denote log-deviations from steady state and

$$\begin{aligned}
\lambda_y &\equiv \sigma + \varphi, \\
\lambda_c &\equiv \frac{\xi(1-\xi)\sigma(1+\lambda_y)}{1+\varphi}, \\
\lambda_\pi &\equiv \frac{\lambda\varepsilon}{(1-\beta\lambda)(1-\lambda)}, \\
\lambda_h &\equiv \sigma_h \xi(1-\xi).
\end{aligned}$$

Proof. Preliminaries: efficient steady state.

Begin by defining the efficient steady state. It maximises

$$U_t \equiv \tilde{\xi} U^b(C_t^b, L_t^b, H_t^b) + (1-\tilde{\xi}) U^s(C_t^s, L_t^s, H_t^s), \tag{29}$$

for some Pareto weights $\tilde{\xi} \in [0,1]$, subject to the aggregate resource constraint:

$$A_t((L_t^b)^\xi (L_t^s)^{1-\xi})^{1-\alpha} = \xi C_t^b + (1-\xi) C_t^s.$$

The Lagrangean for this problem is:

$$\begin{aligned}
\mathcal{L} = &\tilde{\xi} U^b(C_t^b, L_t^b, H_t^b) + (1-\tilde{\xi}) U^s(C_t^s, L_t^s, H_t^s) \\
&+ \lambda \left[A_t((L_t^b)^\xi (L_t^s)^{1-\xi})^{1-\alpha} - \xi C_t^b - (1-\xi) C_t^s \right] \\
&+ \mu [H - \xi H_t^b - (1-\xi) H_t^s]
\end{aligned}$$

In steady state, the first-order conditions are:

$$\tilde{\xi} U_c^b = \lambda \xi$$

$$(1-\tilde{\xi}) U_c^s = \lambda (1-\xi)$$

$$-\tilde{\xi} U_l^b = \lambda (1-\alpha) \xi \frac{Y}{L^b}$$

$$-(1-\tilde{\xi}) U_l^s = \lambda (1-\alpha)(1-\xi) \frac{Y}{L^s}$$

$$\tilde{\xi} U_h^b = \mu \xi$$

$$(1-\tilde{\xi})U_h^s = \mu(1-\xi)$$

These define the 'efficient' steady state.

Loss function.

We approximate welfare around the efficient steady state by taking a second-order approximation of (29). Suppose first that housing demand were fixed. Then:

$$U_t = \tilde{\xi}U^b(C_t^b, L_t^b, H^b) + (1-\tilde{\xi})U^s(C_t^s, L_t^s, H^s).$$

Approximating this, we get:

$$\begin{aligned}U_t - U \simeq\ & \tilde{\xi}(U_c^b(C_t^b - C^b) + \frac{1}{2}U_{cc}^b(C_t^b - C^b)^2) + (1-\tilde{\xi})(U_c^s(C_t^s - C^s) + \frac{1}{2}U_{cc}^s(C_t^s - C^s)^2) \\ & + \tilde{\xi}(U_l^b(L_t^b - L^b) + \frac{1}{2}U_{ll}^b(L_t^b - L^b)^2) + (1-\tilde{\xi})(U_l^s(L_t^s - L^s) + \frac{1}{2}U_{ll}^s(L_t^s - L^s)^2),\end{aligned}$$

Gathering terms:

$$\begin{aligned}U_t - U \simeq\ & \tilde{\xi}U_c^b((C_t^b - C^b) + \frac{1}{2}\frac{U_{cc}^b}{U_c^b}(C_t^b - C^b)^2) + (1-\tilde{\xi})U_c^s((C_t^s - C^s) + \frac{1}{2}\frac{U_{cc}^s}{U_c^s}(C_t^s - C^s)^2) \\ & + \tilde{\xi}U_l^b((L_t^b - L^b) + \frac{1}{2}\frac{U_{ll}^b}{U_l^b}(L_t^b - L^b)^2) + (1-\tilde{\xi})U_l^s((L_t^s - L^s) + \frac{1}{2}\frac{U_{ll}^s}{U_l^s}(L_t^s - L^s)^2),\end{aligned}$$

Using the first-order conditions associated with the efficient steady state, we get

$$\begin{aligned}U_t - U \simeq\ & \lambda\xi((C_t^b - C^b) + \frac{1}{2}\frac{U_{cc}^b}{U_c^b}(C_t^b - C^b)^2) + \lambda(1-\xi)((C_t^s - C^s) + \frac{1}{2}\frac{U_{cc}^s}{U_c^s}(C_t^s - C^s)^2) \\ & - \lambda(1-\alpha)\xi\frac{Y}{L^b}((L_t^b - L^b) + \frac{1}{2}\frac{U_{ll}^b}{U_l^b}(L_t^b - L^b)^2) - \lambda(1-\alpha)(1-\xi)\frac{Y}{L^s}((L_t^s - L^s) \\ & + \frac{1}{2}\frac{U_{ll}^s}{U_l^s}(L_t^s - L^s)^2),\end{aligned}$$

Note that the preferences used above imply that:

$$\begin{aligned}\frac{U_{cc}^j}{U_c^j} &= \frac{-z^2\exp(-zC)}{z\exp(-zC)} = -z = -\frac{\sigma}{Y} \\ \frac{U_{ll}^j}{U_l^j} &= \frac{\chi\varphi(L^j)^{\varphi-1}}{\chi(L^j)^{\varphi}} = \frac{\varphi}{L^j}\end{aligned}$$

So gathering terms in demand and using these, we get:

$$
\begin{aligned}
U_t - U \simeq\ & \lambda \left[\xi(C_t^b - C^b) + (1-\xi)(C_t^s - C^s)\right] - \frac{1}{2}\lambda z \left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right] \\
& - \lambda(1-\alpha)\xi \frac{Y}{L^b}((L_t^b - L^b) + \frac{1}{2}\frac{\varphi}{L^b}(L_t^b - L^b)^2) - \lambda(1-\alpha)(1-\xi)\frac{Y}{L^s}((L_t^s - L^s) \\
& + \frac{1}{2}\frac{\varphi}{L^s}(L_t^s - L^s)^2),
\end{aligned}
$$

Now we use the approximation for first-order terms:

$$
\begin{aligned}
X_t &= X(1 + \frac{\ln X_t}{X} + \frac{1}{2}\left(\frac{\ln X_t}{X}\right)^2 + ...) \\
&= X(1 + x_t + \frac{1}{2}x_t^2 + ...)
\end{aligned}
$$

such that using a second-order approximation to the aggregate resource constraint gets us:

$$
\begin{aligned}
U_t - U \simeq\ & \lambda Y\left(y_t + \frac{1}{2}y_t^2\right) - \frac{1}{2}\lambda z \left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right] \\
& - \lambda(1-\alpha)\xi Y(l_t^b + \frac{1}{2}(1+\varphi)(l_t^b)^2) - \lambda(1-\alpha)(1-\xi)Y(l_t^s + \frac{1}{2}(1+\varphi)(l_t^s)^2),
\end{aligned}
$$

To deal with the terms involving labour, we need to turn to the labour supply conditions. These are, respectively:

$$
\begin{aligned}
\frac{\chi(L_t^b)^{1+\varphi}}{U_{ct}^b} &= W_t^b L_t^b \\
\frac{\chi(L_t^s)^{1+\varphi}}{U_{ct}^s} &= W_t^s L_t^s
\end{aligned}
$$

and the weighted aggregate version is:

$$
\frac{\chi(L_t)^{1+\varphi}}{(U_{ct}^b)^{\xi}(U_{ct}^s)^{1-\xi}} = \frac{\chi(L_t)^{1+\varphi}}{z\exp(-zY_t)} = W_t L_t
$$

The trick will be to write the term involving the aggregate labour index in terms of aggregate output and price dispersion. To do this, we need to use the production function at the firm level:

$$
Y_t(i) = A_t L_t(i)^{1-\alpha} \Rightarrow L_t(i) = \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}},
$$

to write aggregate employment as:

$$
L_t = \int_0^1 L_t(i)di = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di.
$$

We need this together with the firm-level demand function:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t,$$

to write:

$$L_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di = \left(\frac{\Delta_t Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$$

where

$$\Delta_t^{\frac{1}{1-\alpha}} \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di$$

is an index of price dispersion. Using these conditions, we can eliminate the aggregate labour index from the aggregate supply condition, writing it as:

$$\frac{\chi}{z\exp(-zY_t)} \left(\frac{\Delta_t Y_t}{A_t}\right)^{\frac{1+\varphi}{1-\alpha}} = W_t L_t$$

Note that $W_t L_t = W_t^j L_t^j$ (i.e. each type's labour income is proportional to aggregate labour income, via the production function). As a result, we can write both types' labour supply functions in terms of the aggregate as:

$$\frac{\chi (L_t^b)^{1+\varphi}}{z\exp(-zC_t^b)} = \frac{\chi}{z\exp(-zY_t)} \left(\frac{\Delta_t Y_t}{A_t}\right)^{\frac{1+\varphi}{1-\alpha}}$$

$$\frac{\chi (L_t^s)^{1+\varphi}}{z\exp(-zC_t^s)} = \frac{\chi}{z\exp(-zY_t)} \left(\frac{\Delta_t Y_t}{A_t}\right)^{\frac{1+\varphi}{1-\alpha}}$$

Approximating these conditions yields:

$$(1+\varphi) l_t^j + z(C_t^j - C^j) \simeq z(Y_t - Y) + \frac{1+\varphi}{1-\alpha}(d_t + y_t - a_t)$$

where:

$$d_t \equiv (1-\alpha) \log \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di,$$

(which equals zero to first order). Rearranging this gives:

$$l_t^j \simeq \frac{1}{1-\alpha}(d_t + y_t - a_t) - \frac{z}{1+\varphi}\left[(C_t^j - C^j) - (Y_t - Y)\right], \quad j = b, s$$

Using $\sigma \equiv zY$, we can also write this as:

$$l_t^j \simeq \frac{1}{1-\alpha}(d_t + y_t - a_t) - \frac{\sigma}{1+\varphi}(c_t^j - y_t), \quad j = b, s.$$

Using these together with the aggregate resource constraint to first order, $y_t = \xi c_t^b + (1-\xi) c_t^s$, which implies:

$$c_t^s - y_t = -\xi(c_t^b - c_t^s), \quad \text{and} \quad c_t^b - y_t = (1-\xi)(c_t^b - c_t^s)$$

such that:

$$l_t^b \simeq \frac{1}{1-\alpha}(d_t + y_t - a_t) - \frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s)$$
$$l_t^s \simeq \frac{1}{1-\alpha}(d_t + y_t - a_t) + \frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s)$$

We have now written the individual labour supplies in terms of aggregate variables and the borrower-saver consumption gap. Using these expressions in the loss function allows us to write:

$$\begin{aligned}
U_t - U \simeq{}& \lambda Y\left(y_t + \frac{1}{2}y_t^2\right) - \frac{1}{2}\lambda z\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right] \\
&- \lambda(1-\alpha)Y\begin{pmatrix} \xi\frac{1}{1-\alpha}(d_t + y_t - a_t) - \xi\frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s) \\ +(1-\xi)\frac{1}{1-\alpha}(d_t + y_t - a_t) + (1-\xi)\frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s) \end{pmatrix} \\
&- \frac{1}{2}\lambda(1-\alpha)\xi Y(1+\varphi)(l_t^b)^2 - \frac{1}{2}\lambda(1-\alpha)(1-\xi)Y(1+\varphi)(l_t^s)^2,
\end{aligned}$$

Simplifying the term in labour:

$$\begin{aligned}
U_t - U \simeq{}& \frac{1}{2}\lambda Y y_t^2 - \frac{1}{2}\lambda z\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right] \\
&- \lambda Y(d_t - a_t) \\
&- \frac{1}{2}\lambda(1-\alpha)\xi Y(1+\varphi)(l_t^b)^2 - \frac{1}{2}\lambda(1-\alpha)(1-\xi)Y(1+\varphi)(l_t^s)^2,
\end{aligned}$$

gather terms:

$$\begin{aligned}
U_t - U \simeq{}& -\frac{1}{2}\lambda Y\left[-y_t^2 + \frac{z}{Y}\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right]\right] \\
&- \lambda Y(d_t - a_t) \\
&- \frac{1}{2}\lambda(1-\alpha)\xi Y(1+\varphi)(l_t^b)^2 - \frac{1}{2}\lambda(1-\alpha)(1-\xi)Y(1+\varphi)(l_t^s)^2,
\end{aligned}$$

use c_t^j notation:

$$\begin{aligned}
U_t - U \simeq{}& -\frac{1}{2}\lambda Y\left[-y_t^2 + \sigma\left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2\right]\right] \\
&- \lambda Y(d_t - a_t) \\
&- \frac{1}{2}\lambda(1-\alpha)\xi Y(1+\varphi)(l_t^b)^2 - \frac{1}{2}\lambda(1-\alpha)(1-\xi)Y(1+\varphi)(l_t^s)^2,
\end{aligned}$$

gather terms:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[-y_t^2 + \sigma \left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2\right] + (1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2\right]\right]$$
$$- \lambda Y(d_t - a_t),$$

Add and substract $(\varphi + \sigma)y_t^2$:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 - (\varphi+\sigma)y_t^2 - y_t^2 + \sigma\left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2\right] \\ +(1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

take σy_t^2 inside the consumption gap term:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 - (1+\varphi)y_t^2 + \sigma\left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2 - y_t^2\right] \\ +(1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

and take $(1+\varphi)y_t^2$ inside the labour gap term:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{l} (\varphi+\sigma)y_t^2 + \sigma\left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2 - y_t^2\right] \\ +(1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - \frac{1}{1-\alpha}y_t^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

Combine the y_t^2 term inside the consumption gap term with individual types' consumption:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 + \sigma\left[\xi\left((c_t^b)^2 - y_t^2\right) + (1-\xi)\left((c_t^s)^2 - y_t^2\right)\right] \\ +(1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - \frac{1}{1-\alpha}y_t^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

and use

$$(c_t^j)^2 - y_t^2 = (c_t^j + y_t)(c_t^j - y_t)$$

to get:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 + \sigma\left[\xi(c_t^b + y_t)(c_t^b - y_t) + (1-\xi)(c_t^s + y_t)(c_t^s - y_t)\right] \\ +(1-\alpha)(1+\varphi)\left[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - \frac{1}{1-\alpha}y_t^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

which simplifies out to:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{l} (\varphi+\sigma)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)(1+\varphi)\left[\xi\left(l_t^b\right)^2 + (1-\xi)\left(l_t^s\right)^2 - \frac{1}{1-\alpha}y_t^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

where we used the results above to write:

$$\begin{aligned} \sigma\left[\xi(c_t^b + y_t)(c_t^b - y_t) + (1-\xi)(c_t^s + y_t)(c_t^s - y_t)\right] &= \sigma \left[\begin{array}{c} \xi(c_t^b + y_t)(1-\xi)(c_t^b - c_t^s) \\ -\xi(1-\xi)(c_t^s + y_t)(c_t^b - c_t^s) \end{array} \right] \\ &= \sigma\xi(1-\xi)(c_t^b - c_t^s)\left[(c_t^b + y_t) - (c_t^s + y_t)\right] \\ &= \sigma\xi(1-\xi)(c_t^b - c_t^s)^2 \end{aligned}$$

leaving only the labour terms to work on.

Next, use the labour supply functions to eliminate the terms in squared-labour:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{cc} & (\varphi+\sigma)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)(1+\varphi) & \left[\begin{array}{c} \xi\left(\frac{1}{1-\alpha}(d_t + y_t - a_t) - \frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s)\right)^2 \\ +(1-\xi)\left(\frac{1}{1-\alpha}(d_t + y_t - a_t) + \frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s)\right)^2 - \frac{1}{1-\alpha}y_t^2 \end{array} \right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

Multiply these terms out and ignore the higher-order terms that result, giving:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{cc} & (\varphi+\sigma)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)(1+\varphi) & \left[\begin{array}{c} \xi\left(\frac{1}{1-\alpha}\right)^2(d_t + y_t - a_t)^2 + \xi\left(\frac{\sigma}{1+\varphi}(1-\xi)\right)^2(c_t^b - c_t^s)^2 \\ +(1-\xi)\left(\frac{1}{1-\alpha}\right)^2(d_t + y_t - a_t)^2 \\ +(1-\xi)\left(\frac{\sigma}{1+\varphi}\xi\right)^2(c_t^b - c_t^s)^2 - \frac{1}{1-\alpha}y_t^2 \end{array} \right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

consolidate the aggregate terms:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)(1+\varphi)\left[\left(\frac{1}{1-\alpha}\right)^2(d_t + y_t - a_t)^2 + \left(\frac{\sigma}{1+\varphi}\right)^2\xi(1-\xi)(c_t^b - c_t^s)^2 - \frac{1}{1-\alpha}y_t^2\right] \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

multiply out:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\varphi+\sigma)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \\ +\frac{1+\varphi}{1-\alpha}(d_t + y_t - a_t)^2 - (1+\varphi)y_t^2 \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

gather terms:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\sigma-1)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 + \frac{1+\varphi}{1-\alpha}((y_t - a_t) + d_t)^2 \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

multiply out $((y_t - a_t) + d_t)^2$

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\sigma-1)y_t^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 + \frac{1+\varphi}{1-\alpha}((y_t - a_t)^2 + d_t^2 + 2(y_t - a)d_t) \end{array} \right]$$
$$- \lambda Y(d_t - a_t),$$

In this, ignoring higher-order terms and t.i.p:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\sigma-1)y_t^2 + \frac{1+\varphi}{1-\alpha}(y_t - a_t)^2 + \xi(1-\xi)\sigma(c_t^b - c_t^s)^2 \\ +(1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{array} \right]$$
$$- \lambda Y d_t,$$

Expand $(y_t - a_t)^2$

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} (\sigma-1)y_t^2 + \frac{1+\varphi}{1-\alpha}\left(y_t^2 + a_t^2 - 2y_t a_t\right) \\ +\xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{array} \right]$$
$$- \lambda Y d_t,$$

ignore t.i.p:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\begin{array}{c} \left(\sigma - 1 + \frac{1+\varphi}{1-\alpha}\right) y_t^2 - 2\frac{1+\varphi}{1-\alpha}y_t a_t \\ +\xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{array} \right]$$
$$- \lambda Y d_t,$$

manipulate coefficient on y_t^2:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \begin{bmatrix} \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} y_t^2 - 2\frac{1+\varphi}{1-\alpha} y_t a_t \\ +\xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{bmatrix}$$
$$- \lambda Y d_t,$$

Now recall that the natural level of output is:

$$y_t^* = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha} a_t$$

so:

$$a_t = \frac{\sigma(1-\alpha)+\varphi+\alpha}{1+\varphi} y_t^*$$

Using this to eliminate a_t:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \begin{bmatrix} \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} y_t^2 - 2\frac{\sigma(1-\alpha)+\varphi+\alpha}{1-\alpha} y_t y_t^* \\ +\xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{bmatrix}$$
$$- \lambda Y d_t,$$

Gathering terms:

$$U_t - U \simeq -\frac{1}{2}\lambda Y \begin{bmatrix} \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} \left(y_t^2 - 2 y_t y_t^*\right) \\ +\xi(1-\xi)\sigma(c_t^b - c_t^s)^2 + (1-\alpha)\frac{\sigma^2}{1+\varphi}\xi(1-\xi)(c_t^b - c_t^s)^2 \end{bmatrix}$$
$$- \lambda Y d_t,$$

Use that

$$\begin{aligned} (y_t - y_t^*)^2 &= \tilde{y}_t^2 \\ &= y_t^2 - 2y_t y_t^* + (y_t^*)^2 \\ &= y_t^2 - 2y_t y_t^* + \text{t.i.p} \end{aligned}$$

where $\tilde{y}_t^2$ is the output gap. So

$$U_t - U \simeq -\frac{1}{2}\lambda Y \left[\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 + \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2\right] - \lambda Y d_t,$$

We then use standard results to work on the price dispersion term. Gali shows that:

$$d_t = \frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\}, \quad \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}.$$

So

$$U_t - U \simeq -\frac{1}{2}\lambda Y\left[\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 + \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2\right] - \lambda Y\frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\},$$

Dividing through by λY gives

$$\frac{U_t - U}{\lambda Y} \simeq -\frac{1}{2}\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 - \frac{1}{2}\xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 - \frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\},$$

Defining the discounted sum of utilities $\mathcal{W}$ as

$$\begin{aligned}\mathcal{W} &\equiv E_0\sum_{t=0}^{\infty}\beta^t\frac{U_t - U}{\lambda Y} \\ &= E_0\sum_{t=0}^{\infty}\beta^t\left[\begin{array}{c} -\frac{1}{2}\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 - \frac{1}{2}\xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 \\ -\frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\}\end{array}\right]\end{aligned}$$

Finally, use Gali's final Lemma, which states that:

$$E_0\sum_{t=0}^{\infty}\beta^t\text{var}_i\{p_t(i)\} = \frac{\theta}{(1-\beta\theta)(1-\theta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2,$$

to get the welfare function:

$$W \simeq -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 + \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 + \frac{\varepsilon}{\Theta}\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2\right]$$

Under the $\alpha = 0$ special case:

$$W \simeq -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[(\sigma+\varphi)\tilde{y}_t^2 + \xi(1-\xi)\sigma\left(1+\frac{\sigma}{1+\varphi}\right)(c_t^b - c_t^s)^2 + \varepsilon\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2\right]$$

Observe that:

the higher is σ, the greater the weight on the output and consumption gap terms, though the weight on the consumption gap grows quadratically in σ, whereas the weight on the output gap is linear in σ;

the higher is φ, the greater the weight on the aggregate output gap, and the smaller the weight on the consumption gap.

To give a rough idea of magnitudes, take $\sigma = \varphi = 1$ as a baseline case. Then

$$W \simeq -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[2\tilde{y}_t^2 + \xi(1-\xi)\frac{3}{2}(c_t^b - c_t^s)^2 + \varepsilon\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2\right]$$
$$= -E_0\sum_{t=0}^{\infty}\beta^t\left[\tilde{y}_t^2 + \xi(1-\xi)\frac{3}{4}(c_t^b - c_t^s)^2 + \frac{\varepsilon}{2}\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2\right]$$

Since $\xi(1-\xi)$ reaches a maximum of 1/4, so the maximum weight on the consumption gap in this case is (1/4)*(3/4) =3/16, around a quarter of the size of the weight on the output gap. However, the variance of $(c_t^b - c_t^s)^2$ could be large relative to that of y_t^2.

Accounting for portfolio adjustment costs.

In the case in which there are, equity adjustment costs use up some of the final goods in the economy. The aggregate resource constraint becomes:

$$Y_t = \xi C_t^b + (1-\xi)C_t^s + h(\tilde{\kappa}_t),$$

where:

$$h(\tilde{\kappa}_t) \equiv \frac{\Psi}{2}\left(\frac{\tilde{\kappa}_t}{\tilde{\kappa}} - 1\right)^2\tilde{\kappa}K_t,$$

is the functional form used above. It satisfies:

$$h'(\tilde{\kappa}_t) = \Psi\left(\frac{\tilde{\kappa}_t}{\tilde{\kappa}} - 1\right)K_t, \quad h'(\tilde{\kappa}) = 0$$

$$h''(\tilde{\kappa}_t) = \Psi\frac{K_t}{\tilde{\kappa}}, \quad h''(\tilde{\kappa}) = \Psi\frac{K}{\tilde{\kappa}}$$

Using this to approximate the resource constraint gives:

$$\xi(C_t^b - C^b) + (1-\xi)(C_t^s - C^s) \simeq Y\left(y_t + \frac{1}{2}y_t^2\right) - \frac{1}{2}\tilde{\kappa}\Psi K\kappa_t^2$$

Now return to the approximate welfare function:

$$U_t - U \simeq \lambda\left[\xi(C_t^b - C^b) + (1-\xi)(C_t^s - C^s)\right] - \frac{1}{2}\lambda z\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right]$$
$$-\lambda(1-\alpha)\xi\frac{Y}{L^b}((L_t^b - L^b) + \frac{1}{2}\frac{\varphi}{L^b}(L_t^b - L^b)^2) - \lambda(1-\alpha)(1-\xi)\frac{Y}{L^s}((L_t^s - L^s) + \frac{1}{2}\frac{\varphi}{L^s}(L_t^s - L^s)^2),$$

Eliminating the term in household demand, as above, now gives:

$$U_t - U \simeq \lambda Y\left(y_t + \frac{1}{2}y_t^2\right) - \lambda\frac{1}{2}\tilde{\kappa}\Psi K\kappa_t^2 - \frac{1}{2}\lambda z\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right]$$
$$-\lambda(1-\alpha)\xi\frac{Y}{L^b}((L_t^b - L^b) + \frac{1}{2}\frac{\varphi}{L^b}(L_t^b - L^b)^2) - \lambda(1-\alpha)(1-\xi)\frac{Y}{L^s}((L_t^s - L^s) + \frac{1}{2}\frac{\varphi}{L^s}(L_t^s - L^s)^2),$$

We need to carry the additional $-\lambda\frac{1}{2}\tilde{\kappa}\Psi K\kappa_t^2$ term through the remainder of the derivation, which does not change. Following the same steps as above, we eventually get to:

$$\begin{aligned}\frac{U_t - U}{\lambda Y} \simeq & -\frac{1}{2}\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 - \frac{1}{2}\xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 \\ & - \frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\} - \frac{1}{2}\tilde{\kappa}\Psi\frac{K}{Y}\kappa_t^2,\end{aligned}$$

Note that $K/Y = D/Y \equiv \eta$ is the aggregate debt to GDP ratio. And note that above we defined $\psi \equiv \tilde{\kappa}\Psi$ as the parameter governing the impact of capital requirements on the borrowing rate in the bank spread equation ($r^b = r_t^d + \psi\kappa_t$). So

$$\begin{aligned}\frac{U_t - U}{\lambda Y} \simeq & -\frac{1}{2}\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 - \frac{1}{2}\xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 \\ & - \frac{\varepsilon}{2\Theta}\text{var}_i\{p_t(i)\} - \frac{1}{2}\psi\eta\kappa_t^2,\end{aligned}$$

Bringing this together as above, the welfare function is:

$$W \simeq -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 + \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 + \frac{\varepsilon}{\Theta}\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2 + \psi\eta\kappa_t^2\right]$$

where we now have an additional term capturing costs associated with changing equity capital requirements. Alternatively, defining the loss function $\mathcal{L} \equiv 2(1-\beta)W$, we get:

$$\begin{aligned}\mathcal{L} = & -\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\text{var}(\tilde{y}_t) - \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}\text{var}(c_t^b - c_t^s) \\ & - \frac{\varepsilon}{\Theta}\frac{\theta}{(1-\beta\theta)(1-\theta)}\text{var}(\pi_t) - \psi\eta\text{var}(\kappa_t).\end{aligned}$$

We would expect the weight on $\psi\eta$ to be quite small; however, it nonetheless means that closing the 'consumption gap' would not be costless and the macroprudential authority may face a trade-off between enhancing risk-sharing and smoothing equity requirements.

Accounting for variable housing demand.

With variable housing demand, we have:

$$U_t = \tilde{\xi}U^b(C_t^b, L_t^b, H_t^b) + (1-\tilde{\xi})U^s(C_t^s, L_t^s, H_t^s).$$

Approximating this, we get:

$$\begin{aligned}U_t - U \simeq & \tilde{\xi}(U_c^b(C_t^b - C^b) + \frac{1}{2}U_{cc}^b(C_t^b - C^b)^2) + (1-\tilde{\xi})(U_c^s(C_t^s - C^s) + \frac{1}{2}U_{cc}^s(C_t^s - C^s)^2) \\ & + \tilde{\xi}(U_l^b(L_t^b - L^b) + \frac{1}{2}U_{ll}^b(L_t^b - L^b)^2) + (1-\tilde{\xi})(U_l^s(L_t^s - L^s) + \frac{1}{2}U_{ll}^s(L_t^s - L^s)^2) \\ & + \tilde{\xi}(U_h^b(H_t^b - H^b) + \frac{1}{2}U_{hh}^b(H_t^b - H^b)^2) + (1-\tilde{\xi})(U_h^s(H_t^s - H^s) + \frac{1}{2}U_{hh}^s(H_t^s - H^s)^2),\end{aligned}$$

Focus on the housing component of this:

$$\bar{U}_t^h \equiv \tilde{\xi}(U_h^b(H_t^b - H^b) + \frac{1}{2}U_{hh}^b(H_t^b - H^b)^2) + (1-\tilde{\xi})(U_h^s(H_t^s - H^s) + \frac{1}{2}U_{hh}^s(H_t^s - H^s)^2)$$

First note that the efficient steady state has

$$\tilde{\xi}U_h^b = \lambda_2\xi$$

$$(1-\tilde{\xi})U_h^s = \lambda_2(1-\xi)$$

So we write:

$$\begin{aligned}\bar{U}_t^h &= \tilde{\xi}U_h^b((H_t^b - H^b) + \frac{1}{2}\frac{U_{hh}^b}{U_h^b}(H_t^b - H^b)^2) + (1-\tilde{\xi})U_h^s((H_t^s - H^s) + \frac{1}{2}\frac{U_{hh}^s}{U_h^s}(H_t^s - H^s)^2) \\ &= \lambda_2\left[\xi((H_t^b - H^b) + \frac{1}{2}\frac{U_{hh}^b}{U_h^b}(H_t^b - H^b)^2) + (1-\xi)((H_t^s - H^s) + \frac{1}{2}\frac{U_{hh}^s}{U_h^s}(H_t^s - H^s)^2)\right]\end{aligned}$$

To second order,

$$\frac{H_t^j - H^j}{H} = h_t^j + \frac{1}{2}(h_t^j)^2$$

and denote:

$$\frac{U_{hh}^b}{U_h^b}H \equiv -\breve{\sigma}_h$$

then:

$$\begin{aligned}\frac{\bar{U}_t^h}{\lambda_2} &= \xi(\frac{H}{H}(H_t^b - H^b) + \frac{1}{2}\frac{U_{hh}^b}{U_h^b}\frac{H^2}{H^2}(H_t^b - H^b)^2) + (1-\xi)(\frac{H}{H}(H_t^s - H^s) + \frac{1}{2}\frac{U_{hh}^s}{U_h^s}\frac{H^2}{H^2}(H_t^s - H^s)^2) \\ &= \xi H(h_t^b + \frac{1-\breve{\sigma}_h}{2}(h_t^b)^2) + (1-\xi)H(h_t^s + \frac{1-\breve{\sigma}_h}{2}(h_t^s)^2) \\ &= H\left(\xi h_t^b + (1-\xi)h_t^s\right) + H\frac{1-\breve{\sigma}_h}{2}\left(\xi(h_t^b)^2 + (1-\xi)(h_t^s)^2\right)\end{aligned}$$

To simplify these terms, we now work with the housing constraint. First, recall:

$$H = \xi H_t^b + (1-\xi)H_t^s$$

A first order approximation gives:

$$0 = \xi\left(H_t^b - H^b\right) + (1-\xi)(H_t^s - H^s)$$

Dividing both sides by H, gives

$$0 = \xi\frac{(H_t^b - H^b)}{H} + (1-\xi)\frac{(H_t^s - H^s)}{H}$$

and defining:

$$h_t^j \equiv \frac{H_t^j - H^j}{H}$$

gives:

$$0 = \xi h_t^b + (1-\xi) h_t^s.$$

Squaring this gives:

$$0 = \xi^2 \left(h_t^b\right)^2 + (1-\xi)^2 \left(h_t^s\right)^2 + 2\xi(1-\xi) h_t^b h_t^s$$

and dividing by 2 gives:

$$0 = \frac{1}{2}\xi^2 \left(h_t^b\right)^2 + \frac{1}{2}(1-\xi)^2 \left(h_t^s\right)^2 + \xi(1-\xi) h_t^b h_t^s$$

Second, a second-order approximation gives:

$$0 = \xi \left(h_t^b + \frac{1}{2}\left(h_t^b\right)^2\right) + (1-\xi)\left(h_t^s + \frac{1}{2}\left(h_t^s\right)^2\right) \tag{30}$$

Equating these two gives:

$$\frac{1}{2}\xi^2 \left(h_t^b\right)^2 + \frac{1}{2}(1-\xi)^2 \left(h_t^s\right)^2 + \xi(1-\xi) h_t^b h_t^s = \xi \left(h_t^b + \frac{1}{2}\left(h_t^b\right)^2\right) + (1-\xi)\left(h_t^s + \frac{1}{2}\left(h_t^s\right)^2\right)$$

$$0 = \left[\xi h_t^b + (1-\xi) h_t^s\right] + \frac{1}{2}\left[\xi(1-\xi)\left(h_t^b\right)^2 + (1-\xi)(1-(1-\xi))\left(h_t^s\right)^2 - 2\xi(1-\xi) h_t^b h_t^s\right]$$

This factorises to:

$$\xi h_t^b + (1-\xi) h_t^s = -\frac{1}{2}\xi(1-\xi)(h_t^b - h_t^s)^2$$

giving the housing gap term. Note also that, from (30),

$$\xi \left(h_t^b\right)^2 + (1-\xi)\left(h_t^s\right)^2 = -2\left(\xi h_t^b + (1-\xi) h_t^s\right)$$

So we can write using ():

$$\xi \left(h_t^b\right)^2 + (1-\xi)\left(h_t^s\right)^2 = \xi(1-\xi)(h_t^b - h_t^s)^2$$

Using these in the loss function gives:

$$\begin{aligned} \frac{\bar{U}_t^h}{\lambda_2 H} &= \xi h_t^b + (1-\xi) h_t^s + \frac{1-\breve{\sigma}_h}{2}\left(\xi (h_t^b)^2 + (1-\xi)(h_t^s)^2\right) \\ &= -\frac{\breve{\sigma}_h}{2}\xi(1-\xi)(h_t^b - h_t^s)^2 \end{aligned}$$

or:

$$\bar{U}_t^h = -\lambda_2 H \frac{\breve{\sigma}_h}{2}\xi(1-\xi)(h_t^b - h_t^s)^2$$

Note that in combining this with the remaining terms in the welfare function, we need to divide through by λY, so the term relevant to the loss function is:

$$-\frac{\lambda_2 H}{\lambda_1 Y}\frac{\breve{\sigma}_h}{2}\xi(1-\xi)(h_t^b - h_t^s)^2$$

In the efficiency conditions

$$\frac{\tilde{\xi}}{\xi} = \frac{\lambda_1}{U_c^b} = \frac{\lambda_2}{U_h^b}$$

so

$$\frac{U_h^b}{U_c^b} = \frac{\lambda_2}{\lambda_1}$$

so

$$\frac{\lambda_2 H}{\lambda_1 Y} = \frac{U_h^b}{U_c^b}\frac{H}{Y}$$

Let us pick the ratio of these marginal utilities, by choice of χ_h^j, so that this term is unity. Bringing it together, we get:

$$W \simeq -\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{l} \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t^2 + \xi(1-\xi)\sigma\frac{1+\varphi+\sigma(1-\alpha)}{1+\varphi}(c_t^b - c_t^s)^2 \\ +\frac{\varepsilon}{\Theta}\frac{\theta}{(1-\beta\theta)(1-\theta)}\pi_t^2 + \psi\eta\kappa_t^2 + \breve{\sigma}_h\xi(1-\xi)\left(h_t^b - h_t^c\right)^2 \end{array} \right].$$

This completes the derivation. ■

D Linearised Constraints

In this section, we derive a first-order approximation of the equilibrium conditions that constitute the constraints for the optimal policy problem in our linear-quadratic setting. Unless otherwise stated, lower-case variables denote log-deviations from steady state, that is, for a generic variable X_t with steady state value Z, $Z_t \equiv \ln(Z_t/Z)$.

D.1 Savers

The Euler equation for savers is

$$c_t^s = -\sigma^{-1}(i_t - \mathbb{E}_t\pi_{t+1}) + \mathbb{E}_t c_{t+1}^s + u_t^c, \tag{31}$$

where $i_t \equiv \ln R_t^d$ is the net nominal interest rate on deposits, $\sigma \equiv zY$ is the inverse of the elasticity of intertemporal substitution, and $u_t^c = \rho_c u_{t-1}^c + \epsilon_t^c$ is an aggregate demand shock that also affects the borrowers' Euler equation, with $\rho_c \in (0,1)$ and $\epsilon_t^c \sim \mathcal{N}(0,\sigma_c^2)$. A similar condition applies to equity investment, taking into account portfolio adjustment costs. Up to the first order, no arbitrage therefore implies

$$i_t^e = i_t + \Psi\kappa_t. \tag{32}$$

The labour supply condition for savers is

$$w_t^s = \varphi l_t^s + \sigma c_t^s,$$

where w_t^s is the log-deviation of the savers' real wage.

Demand for housing by savers is

$$q_t = \frac{1+\tau^h-\beta_s}{1+\tau^h}(-\sigma_h h_t^s + \sigma c_t^s) + \frac{\beta_s}{1+\tau^h}\mathbb{E}_t(-\sigma c_{t+1}^s + \sigma c_t^s + q_{t+1}) + u_t^h, \tag{33}$$

where $\tau^h = \beta_s - \beta_b - \tilde{\mu}\Theta$ (with $\tilde{\mu} = 1 - \beta_b/\beta_s$) is a tax that makes the steady state allocation of housing efficient, q_t is the log-deviation of the real price of housing from its steady state value, and $u_t^h = \rho_h u_{t-1}^h + \epsilon_t^h$ is a housing demand shock, with $\rho^h \in (0,1)$ and $\epsilon_t^h \sim \mathcal{N}(0, \sigma_h^2)$.

D.2 Borrowers

The Euler equation for borrowers takes into account the effect of the collateral constraint

$$c_t^b = -\sigma^{-1}\left(i_t^b - \mathbb{E}_t\pi_{t+1} + \frac{\tilde{\mu}}{1-\tilde{\mu}}\mu_t\right) + \mathbb{E}_t c_{t+1}^b + u_t^c, \tag{34}$$

where $i_t^b \equiv \ln R_t^b$ is the net nominal interest rate faced by borrowers. Note that, everything else equal, a tightening of the collateral constraint ($\mu_t > 0$) tends to raise the cost of borrowing for impatient households.

Labour supply for borrowers is

$$w_t^b = \varphi l_t^b + \sigma c_t^b,$$

where w_t^b is the log-deviation of the borrowers' real wage.

Borrowers' demand for housing is

$$q_t = \frac{\tilde{\mu}\Theta}{1-\tilde{\mu}\Theta}(\mu_t + \theta_t) + \frac{1-\tilde{\mu}\Theta-\beta_b}{1-\tilde{\mu}\Theta}(-\sigma_h h_t^b + \sigma c_t^b) + \frac{\beta_b}{1-\tilde{\mu}\Theta}\mathbb{E}_t(-\sigma c_{t+1}^b + \sigma c_t^b + q_{t+1}) + u_t^h, \tag{35}$$

where θ_t, which can be either a shock or a macro-prudential policy instrument, is the log-deviation of the collateral constraint parameter (the LTV ratio) from its steady state value.

The linearised borrowing constraint at equality is

$$d_t^b = \theta_t + q_t + h_t^b, \tag{36}$$

where d_t^b denotes the log-deviation of the real quantity of debt from its steady state value.

Finally, from the borrowers' budget constraint, we can derive the law of motion for debt as

$$d_t^b = R^b(i_{t-1}^b + d_{t-1}^b - \pi_t) + \frac{1}{\Theta}(h_t^b - h_{t-1}^b) + \frac{1}{\eta}(c_t^b - y_t), \tag{37}$$

where $\eta \equiv (D^b/P)/Y$ represents the steady state real household debt-to-GDP ratio.

D.3 Banks

Banks price loans as a weighted average between the return on equity and the deposit rate

$$i_t^b = \tilde{\kappa} i_t^e + (1 - \tilde{\kappa}) i_t.$$

Using the no arbitrage condition between return on equity and on deposits from the savers' problem (32), we obtain an expression for the spread of the the loan rate on the deposit rate

$$i_t^b = i_t + \psi \kappa_t, \tag{38}$$

where $\psi \equiv \xi \tilde{\kappa} \Psi / (1 - \xi)$ and κ_t is the log-deviation of the capital requirement from its steady state value, which, like θ_t, can be either a shock or a macro-prudential policy instrument.

D.4 Production

Up to a linear approximation, the production function is

$$y_t = a_t + l_t.$$

The labour demand equation is

$$w_t = m_t + y_t - l_t,$$

where m_t is the real marginal cost. The wage bill must be equal across types

$$w_t^s + l_t^s = w_t^b + l_t^b,$$

where the wage index is

$$w_t = \xi w_t^b + (1 - \xi) w_t^s$$

and labour market clearing requires

$$l_t = \xi l_t^b + (1 - \xi) l_t^s.$$

Finally, the Phillips Curve is

$$\pi_t = \frac{(1 - \lambda)(1 - \beta\lambda)}{\lambda} m_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^m,$$

where $u_t^m = \rho_m u_{t-1}^m + \epsilon_t^m$ is a mark-up shock, with $\rho_m \in (0, 1)$ and $\epsilon_t^m \sim \mathcal{N}(0, \sigma_m^2)$.

D.5 Market Clearing

Goods market clearing entails

$$y_t = \xi c_t^b + (1-\xi)c_t^s, \tag{39}$$

while housing market clearing requires

$$\xi h_t^b + (1-\xi)h_t^s = 0. \tag{40}$$

The market clearing conditions complete the description of the linearised model.

D.6 Gaps and Aggregate Variables

In what follows, we combine the equilibrium relations to obtain a parsimonious set of constraints for the optimal policy problem. To simplify the derivations, we assume $\Theta = 1$ (a 100% LTV ratio). We return to the case $\Theta < 1$ in the quantitative analysis.

On the supply side, we can rewrite the Phillips curve in terms of the efficient output gap by noting that, with flexible prices and no markup shocks, $m_t = 0$. Weighting the labour supply equations by population shares, we can write the equilibrium in the labour market as

$$l_t^* + \sigma c_t^* = a_t,$$

where an * represents variables in the efficient equilibrium. Using the production function and the resource constraint, we can solve for the efficient level of output

$$y_t^* = \frac{1+\varphi}{\sigma+\varphi} a_t.$$

With sticky prices, the labour market equilibrium condition, expressed in terms of output, is

$$m_t = (\sigma+\varphi)y_t - (1+\varphi)a_t = (\sigma+\varphi)(y_t - y_t^*).$$

Replacing into the Phillips curve gives

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^m, \tag{41}$$

where $\gamma \equiv (\sigma+\varphi)(1-\lambda)(1-\beta\lambda)/\lambda$ and $x_t \equiv y_t - y_t^*$ is the efficient output gap.

On the demand side, we start from the savers' Euler equation (31) and replace savers' consumption from the resource constraint (39) to obtain

$$y_t - \xi(c_t^b - c_t^s) = -\sigma^{-1}(i_t - \mathbb{E}_t\pi_{t+1}) + \mathbb{E}_t[y_{t+1} - \xi(c_{t+1}^b - c_{t+1}^s)] + u_t^c.$$

We can express the last equation in terms of output gap

$$x_t - \xi(c_t^b - c_t^s) = -\sigma^{-1}(i_t - \mathbb{E}_t\pi_{t+1}) + \mathbb{E}_t[x_{t+1} - \xi(c_{t+1}^b - c_{t+1}^s)] + \nu_t^{cgap}, \tag{42}$$

where

$$\nu_t^{cgap} \equiv u_t^c + \mathbb{E}_t y_{t+1}^* - y_t^*.$$

The second condition that characterises aggregate demand comes from substituting housing demand for borrowers from the market clearing condition (40) into the borrowing constraint equality (36)

$$d_t^b = \theta_t + q_t + (1-\xi)(h_t^b - h_t^s). \tag{43}$$

Finally, we can replace the goods and housing market resource constraints, and the banking condition (38) into the borrowers' budget constraint (37) to obtain an equation for the law of motion of debt

$$d_t^b = \frac{1}{\beta_s}(i_{t-1} + \psi\kappa_{t-1} + d_{t-1}^b - \pi_t) + (1-\xi)[(h_t^b - h_t^s) - (h_{t-1}^b - h_{t-1}^s)] + \frac{1-\xi}{\eta}(c_t^b - c_t^s). \tag{44}$$

For the housing gap, we take the difference of the two housing demand functions and eliminate the multiplier on the borrowing constraint from borrowers' Euler equation to obtain

$$\begin{aligned}(1-\tilde{\mu}-\beta_b)\sigma_h(h_t^b - h_t^s) &= -\sigma(1-\tilde{\mu})\mathbb{E}_t(c_t^b - c_{t+1}^b) - (1-\tilde{\mu})(i_t^{b*} - \mathbb{E}_t\pi_{t+1} + u_t^c) + \tilde{\mu}\theta_t \\ &+ (1-\tilde{\mu}-\beta_b)\sigma(c_t^b - c_t^s) + (\beta_s - \beta_b)(q_t - u_t^h - \mathbb{E}_t q_{t+1}) + \beta_b\mathbb{E}_t\sigma(c_t^b - c_{t+1}^b) - \beta_s\mathbb{E}_t\sigma(c_t^s - c_{t+1}^s).\end{aligned}$$

We can use the savers' Euler equation to eliminate the last term of the previous expression, as well as the banks' zero profit condition to get rid of the borrowing rate. After substituting for borrowers' consumption from the resource constraint and using again the savers' Euler equation, some manipulations allow us to write

$$\begin{aligned}h_t^b - h_t^s = &-\frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h\xi\omega}(i_t - \mathbb{E}_t\pi_{t+1}) + \frac{\beta_s - \beta_b}{\sigma_h\omega}(q_t - \mathbb{E}_t q_{t+1}) - \frac{\sigma}{\sigma_h\xi}(x_t - \mathbb{E}_t x_{t+1}) \\ &+ \frac{\sigma}{\sigma_h}(c_t^b - c_t^s) + \frac{\tilde{\mu}}{\sigma_h\omega}\theta_t - \frac{1-\tilde{\mu}}{\sigma_h\omega}\psi\kappa_t + \nu_t^{hgap},\end{aligned} \tag{45}$$

where

$$\nu_t^{hgap} \equiv -\frac{\sigma}{\sigma_h\xi}(y_t^* - \mathbb{E}_t y_{t+1}^*) - \frac{\beta_s - \beta_b}{\sigma_h\omega}u_t^h + \frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h\xi\omega}\sigma u_t^c,$$

and $\omega \equiv 1 - \tilde{\mu} - \beta_b = \beta_b(1/\beta_s - 1) > 0$.

To complete the description of the demand side, we start by taking a population-weighted average of the two housing demand equations. We then use the goods and housing market equilibrium conditions to get

$$(\omega+\beta)(q_t - u_t^h) = \xi\tilde{\mu}(\mu_t + \theta_t) + \sigma\omega y_t + \xi\beta_b[\sigma(c_t^b - \mathbb{E}_t c_{t+1}^b) + \mathbb{E}_t q_{t+1}] + (1-\xi)\beta_s[\sigma(c_t^s - \mathbb{E}_t c_{t+1}^s) + \mathbb{E}_t q_{t+1}].$$

If we use the borrowers' Euler equation to eliminate the Lagrange multiplier, replace borrowers' consumption from the goods market equilibrium, savers' consumption from their Euler equation, and the borrowing rate from the banks' zero profit condition, we obtain an aggregate house price equation that reads as

$$q_t = -(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\sigma\omega}{\omega+\beta}\mathbb{E}_t x_{t+1} + \frac{\xi\tilde{\mu}}{\omega+\beta}\theta_t - \frac{\xi(1-\tilde{\mu})}{\omega+\beta}\psi\kappa_t + \frac{\beta}{\omega+\beta}\mathbb{E}_t q_{t+1} + \nu_t^h \tag{46}$$

where

$$\nu_t^h \equiv u_t^h + \sigma u_t^c + \frac{\sigma\omega}{\omega+\beta}\mathbb{E}_t y_{t+1}^*.$$

Given a specification of monetary and macro-prudential policy $\{i_t, \theta_t, \kappa_t\}$, equations (41)-(46), constitute a system of six equations in six unknowns (x_t, π_t, $c_t^b - c_t^s$, $h_t^b - h_t^s$, q_t, and d_t^b) that characterises the equilibrium.

E Optimal Macro-Prudential Policy

In this section, we study optimal macro-prudential policy in the linear-quadratic approximation that we have derived so far. We begin with the case of flexible prices and no cost of changing capital requirements.

E.1 Efficient Equilibrium

In an efficient equilibrium, prices are flexible and there are no markup shocks ($u_t^m = 0$, $\forall t$). In addition, to begin, we also assume no costs of varying the capital requirements ($\lambda_\kappa = 0$). Then, from the Phillips curve, we have $x_t = 0$, that is, output is solely determined by productivity. Moreover, inflation disappears from the loss function because $\lambda_\pi = 0$. In this section we ask if, in this equilibrium, the policymaker can close at the same time the consumption and housing gaps.

Notice that we can characterise the aggregate demand curve, the housing gap equation, and the house price equation in terms of the real interest rate $r_t \equiv i_t - \mathbb{E}_t \pi_{t+1}$. In particular, from (42), a zero consumption and output gap imply $r_t = \sigma v_t^{cgap}$.

In order to derive a solution for house prices, we can subtract from (46) from (45) after imposing that all gaps are zero. The result is

$$q_t = \frac{\beta_s}{\omega+\beta_s}(\mathbb{E}_t q_{t+1} - r_t) + \frac{\omega+\beta}{\omega+\beta_s}\nu_t^h - \frac{\xi\sigma_h\omega}{\omega+\beta_s}\nu_t^{hgap}.$$

Substituting from the expressions of the shocks, including the interest rate, we obtain

$$q_t = \sigma\left(y_t^* - \frac{\beta_s}{\omega+\beta_s}\mathbb{E}_t y_{t+1}^*\right) + u_t^h + \frac{\beta_s}{\omega+\beta_s}\mathbb{E}_t q_{t+1}.$$

Assuming productivity and the house price shock follow an AR(1) process with persistence ρ_a and

ρ_h, respectively, we can write the solution for house prices as

$$q_t = \frac{\sigma\omega + \beta_s(\sigma - \rho_a)}{\omega + \beta_s(1 - \rho_a)} y_t^* + \frac{\omega + \beta_s}{\omega + \beta_s(1 - \rho_h)} u_t^h.$$

Next, we go back to the house price equation and solve it in terms of the capital requirement

$$\xi(1 - \tilde{\mu})\psi\kappa_t = -(\omega + \beta)r_t + \xi\tilde{\mu}\theta_t - (\omega + \beta)q_t + \beta\mathbb{E}_t q_{t+1} + (\omega + \beta)\nu_t^h.$$

Because we have solved for house prices and the interest rate in terms of the exogenous shock, we rewrite the previous expression simply as

$$\psi\kappa_t = \frac{\tilde{\mu}}{1 - \tilde{\mu}}\theta_t + \nu_t^\kappa,$$

where ν_t^κ summarises the shocks that determine the solution for house prices and the interest rate. We replace this expression, together with the expression for the LTV ratio $\theta_t = d_t^b - q_t$, in equation (44) after imposing the housing gap and the consumption gap are zero in all periods

$$d_t^b = \frac{1}{\beta_s}\left[i_{t-1} + \left(1 + \frac{\tilde{\mu}}{1 - \tilde{\mu}}\right) d_{t-1}^b - \pi_t + \nu_t^\kappa\right].$$

The key is to note that we can use the definition of the real interest rate to rewrite the last expression as

$$d_t^b = \frac{1}{\beta_s}\left[\frac{1}{1 - \tilde{\mu}} d_{t-1}^b - (\pi_t - \mathbb{E}_{t-1}\pi_t)\right] + \nu_t^b,$$

where ν_t^b collects all the exogenous shocks that affect the dynamics of debt. In the absence of inflation surprises, any exogenous shock would put debt on an unstable path.[22] Optimal policy therefore requires ex-post inflation surprises that wipe out the effects of the exogenous shocks on debt. The combination of zero consumption and housing gaps is feasible because inflation volatility is costless.

This result extends to the case in which capital requirements are costly to change ($\lambda_\kappa > 0$). In this environment, the policymaker seeks to minimise

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\lambda_c(c_t^b - c_t^s)^2 + \lambda_h(h_t^b - h_t^s)^2 + \lambda_\kappa \kappa_t^2\right] \right\} \tag{47}$$

subject to (42), (43), (45), and (46), with $x_t = 0$. Equation (44) is not a constraint to the optimal policy problem because, again, the policymaker can choose ex-post inflation surprises to stabilise the effect of exogenous shocks on its dynamics and make private debt state-contingent.

Let φ_{nt}, with $n = \{1, 2, 3, 4\}$, be the Lagrange multipliers on the four constraints. Under discretion, the first order conditions for this problem are:

[22] The denominator of the coefficient on lagged debt is $\beta_s(1 - \tilde{\mu}) = \beta_b < 1$.

- For the consumption gap

$$\lambda_c(c_t^b - c_t^s) - \xi\varphi_{1t} + \frac{\sigma}{\sigma_h}\varphi_{2t} = 0, \tag{48}$$

- For the housing gap

$$\lambda_h(h_t^b - h_t^s) - \varphi_{2t} + (1-\xi)\varphi_{4t} = 0, \tag{49}$$

- For the capital requirement

$$\lambda_\kappa\kappa_t - \frac{1-\tilde{\mu}}{\sigma_h\omega}\psi\varphi_{2t} - \frac{\xi(1-\tilde{\mu})}{\omega+\beta}\psi\varphi_{3t} = 0, \tag{50}$$

- For the LTV ratio

$$\frac{\tilde{\mu}}{\sigma_h\omega}\varphi_{2t} + \frac{\xi\tilde{\mu}}{\omega+\beta}\varphi_{3t} + \varphi_{4t} = 0, \tag{51}$$

- For the real interest rate[23]

$$-\sigma^{-1}\varphi_{1t} - \frac{\omega - \xi(\beta_s - \beta_b)}{\xi\sigma_h\omega}\varphi_{2t} - \varphi_{3t} = 0, \tag{52}$$

- For house prices

$$\frac{\beta_s - \beta_b}{\sigma_h\omega}\varphi_{2t} - \varphi_{3t} + \varphi_{4t} = 0. \tag{53}$$

We start by combining the first order conditions for the two macro-prudential instruments (50) and (51) to obtain

$$\varphi_{4t} = -\frac{\tilde{\mu}}{1-\tilde{\mu}}\frac{\lambda_\kappa}{\psi}\kappa_t.$$

We then replace the solution in the the first order condition for the housing gap (49) to get

$$\varphi_{2t} = \frac{1}{1-\xi}\lambda_h(h_t^b - h_t^s) - \frac{\tilde{\mu}}{(1-\xi)(1-\tilde{\mu})}\frac{\lambda_\kappa}{\psi}\kappa_t.$$

Replacing this result back into the first order condition for capital requirements (50), we have

$$\varphi_{3t} = \frac{\omega+\beta}{\xi(1-\tilde{\mu})}\left\{\left[1 + \frac{\tilde{\mu}}{(1-\xi)\sigma_h\omega}\right]\frac{\lambda_\kappa}{\psi}\kappa_t - \frac{1-\tilde{\mu}}{(1-\xi)\sigma_h\omega}\lambda_h(h_t^b - h_t^s)\right\}.$$

Finally, replacing the last two expressions in the first order conditions for the real interest rate (52), we can write

$$\varphi_{1t} = \frac{\beta_s\sigma}{\xi\sigma_h\omega}\lambda_h(h_t^b - h_t^s) - \frac{\sigma}{1-\tilde{\mu}}\left[\frac{(1-\xi)\sigma_h\omega(\omega+\beta) + \beta_s\tilde{\mu}}{\xi(1-\xi)\sigma_h\omega}\right]\frac{\lambda_\kappa}{\psi}\kappa_t.$$

Given the solution for the four Lagrange multipliers, we can use the last two first order conditions

[23] Under flexible prices, inflation enters independently only in the equation for debt, which, as discussed above, is residual, and serves to pin down the inflation surprise necessary to stabilise debt. Therefore, we can express the equilibrium as a function of the real interest rate only.

(48) and (53) to derive the two targeting rules that determine the optimal macro-prudential policy. In particular, the first rule (derived from 53) suggests the policymaker should optimally set capital requirements in response to variations of the housing gap

$$\lambda_\kappa \kappa_t = \varphi_\kappa \lambda_h (h_t^b - h_t^s), \tag{54}$$

where

$$\varphi_\kappa \equiv \frac{\beta_b \psi}{\tilde{\mu}(\omega + \beta_s) + (1-\xi)\sigma_h \omega(\omega + \beta + \xi\tilde{\mu})} > 0.$$

The targeting rule requires an increase in capital requirements whenever the an housing gap opens up (and vice versa).

The second rule (derived from 48) suggests the policymaker should also be concerned with the consequences capital requirements and the housing gap implied by (54) have on the consumption gap

$$\lambda_c(c_t^b - c_t^s) - \frac{\sigma[\tilde{\mu}(\omega + \beta_s) + (1-\xi)\sigma_h \omega(\omega + \beta)]}{(1-\xi)(1-\tilde{\mu})\sigma_h \omega} \frac{\lambda_\kappa}{\psi} \kappa_t + \frac{\sigma[\omega - (1-\xi)\beta_s]}{(1-\xi)\sigma_h \omega} \lambda_h (h_t^b - h_t^s) = 0.$$

Combining the last expression with the targeting rule for capital requirements (54), we obtain an alternative targeting rule that only involves the housing gap and the consumption gap

$$\lambda_c(c_t^b - c_t^s) - \varphi_h \lambda_h (h_t^b - h_t^s) = 0, \tag{55}$$

where

$$\varphi_h \equiv \frac{\sigma\{(1-\xi)\beta_s + \varphi_\kappa[\tilde{\mu}(\omega + \beta_s) + (1-\xi)\sigma_h \omega(\omega + \beta)] - \omega\}}{(1-\xi)\sigma_h \omega}.$$

Since ω is positive but quite small, φ is likely to be positive for most parameter configurations. Therefore, the second targeting rule calls for the policymakers to set the maximum LTV ratio so that the consumption and housing gaps move in the same direction. Using the borrowing constraint at equality, we can indeed express the rule in terms of the instrument

$$\theta_t = d_t^b - q_t - \frac{(1-\xi)\lambda_c}{\varphi_h \lambda_h}(c_t^b - c_t^s).$$

Higher house prices and a positive consumption gap require a tightening of the LTV ratio. Against the common wisdom, the instrument rule suggests LTVs should be relaxed in the face of higher debt, for given house prices and consumption gap. While this result may appear counterintuitive, recall that under optimal policy ex-post inflation surprises make debt state-contingent, hence effectively de-emphasising its relevance in this context.

E.2 Sticky Prices

With sticky prices, the policymaker seeks to minimise

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[x_t^2 + \lambda_\pi \pi_t^2 + \lambda_c (c_t^b - c_t^s)^2 + \lambda_h (h_t^b - h_t^s)^2 + \lambda_\kappa \kappa_t^2 \right] \right\} \tag{56}$$

subject to (41), (42), (44), (43), (45), and (46). Let φ_{nt}, with $n = \{1, 2, 3, 4\}$, be the Lagrange multipliers on the four constraints present in the problem with flexible prices of the previous section, φ_{5t} be the multiplier on (44), and φ_{6t} be the multiplier on (41). The first order conditions for this problem are:

- For the consumption gap

$$\lambda_c (c_t^b - c_t^s) - \xi \varphi_{1t} + \frac{\sigma}{\sigma_h} \varphi_{2t} + \frac{1-\xi}{\eta} \varphi_{5t} = 0, \tag{57}$$

- For the housing gap

$$\lambda_h (h_t^b - h_t^s) - \varphi_{2t} + (1-\xi)\varphi_{4t} + (1\xi)(\varphi_{5t} - \beta \mathbb{E}_t \varphi_{5t+1}) = 0, \tag{58}$$

- For the capital requirement

$$\lambda_\kappa \kappa_t - \frac{1-\tilde{\mu}}{\sigma_h \omega} \psi \varphi_{2t} - \frac{\xi(1-\tilde{\mu})}{\omega + \beta} \psi \varphi_{3t} + \psi \beta \mathbb{E}_t \varphi_{5t+1} = 0, \tag{59}$$

- For the LTV ratio

$$\frac{\tilde{\mu}}{\sigma_h \omega} \varphi_{2t} + \frac{\xi \tilde{\mu}}{\omega + \beta} \varphi_{3t} + \varphi_{4t} = 0, \tag{60}$$

- For the nominal interest rate

$$-\sigma^{-1} \varphi_{1t} - \frac{\omega - \xi(\beta_s - \beta_b)}{\xi \sigma_h \omega} \varphi_{2t} - \varphi_{3t} + \mathbb{E}_t \varphi_{5t+1} = 0, \tag{61}$$

- For house prices

$$\frac{\beta_s - \beta_b}{\sigma_h \omega} \varphi_{2t} - \varphi_{3t} + \varphi_{4t} = 0, \tag{62}$$

- For debt

$$-\varphi_{5t} + \beta \mathbb{E}_t \varphi_{5t+1} - \varphi_{4t} = 0, \tag{63}$$

- For the output gap

$$x_t - \varphi_{1t} - \frac{\sigma}{\sigma_h} \varphi_{2t} + \gamma \varphi_{6t} = 0, \tag{64}$$

- For inflation

$$\lambda_\pi \pi_t - \varphi_{5t} - \varphi_{6t} = 0. \tag{65}$$

Replacing (63) into (58), we can immediately see that

$$\varphi_{2t} = \lambda_h(h_t^b - h_t^s).$$

Subtracting (60) from (62), we obtain

$$\varphi_{3t} = -\frac{\omega + \beta}{\omega + \beta + \xi\tilde{\mu}} \frac{\sigma_h\omega - \beta_s + \beta_b}{\sigma_h\omega} \lambda_h(h_t^b - h_t^s).$$

Plugging back into (60) yields

$$\varphi_{4t} = -\frac{\tilde{\mu}}{\sigma_h\omega} \left[\frac{\omega + \beta_s + \xi(\tilde{\mu} - \sigma_h\omega)}{\omega + \beta + \xi\tilde{\mu}}\right] \lambda_h(h_t^b - h_t^s).$$

Next, we take (61), multiply by ψ, and subtract from (59) to get

$$\varphi_{1t} = -\frac{\sigma\lambda_\kappa}{\psi}\kappa_t - \frac{\sigma}{\sigma_h\omega}\left[\frac{(1-\xi)\omega - \xi\beta_s}{\xi} - \frac{(1-\xi)(\omega+\beta_s)(\sigma_h\omega - \beta_s + \beta_b)}{\omega + \beta + \xi\tilde{\mu}}\right] \lambda_h(h_t^b - h_t^s).$$

If we multiply the (65) by $(1-\xi)/\eta$ and add the result to (57), we can write

$$\varphi_{6t} = \lambda_\pi \pi_t + \frac{\eta}{1-\xi}\left[\lambda_c(c_t^b - c_t^s) + \frac{\sigma}{\sigma_h}\lambda_h(h_t^b - h_t^s) + \xi\varphi_{1t}\right].$$

Finally, replacing back into the equation for (65), we have

$$\varphi_{5t} = -\frac{\eta}{1-\xi}\left[\lambda_c(c_t^b - c_t^s) + \frac{\sigma}{\sigma_h}\lambda_h(h_t^b - h_t^s) + \xi\varphi_{1t}\right].$$

As in the case of flexible prices, given the solution for the Lagrange multipliers as a function of the endogenous variables, we can solve for three targeting rules that characterise optimal policy. We obtain the first substituting the multipliers in (64)

$$x_t + \gamma\lambda_\pi\pi_t + \frac{\eta}{1-\xi}\lambda_c(c_t^b - c_t^s) + \left(1 - \frac{\xi\eta}{1-\xi}\right)\left[\frac{\sigma}{\psi}\lambda_\kappa\kappa_t - \alpha_h\lambda_h(h_t^b - h_t^s)\right] = 0, \tag{66}$$

where

$$\alpha_h \equiv \frac{\sigma(\omega+\beta_s)[1 + (1-\xi)(\sigma_h\omega - \tilde{\mu})]}{\sigma_h\omega(\omega + \beta + \xi\tilde{\mu})}$$

The first two terms correspond to the optimal targeting rule for monetary policy under discretion in the absence of heterogeneity and financial frictions (i.e., in the standard New Keynesian model). The next three terms reveal how the presence of imperfect risk sharing affects the conduct of optimal monetary policy.

The second targeting rule follows from substituting the multipliers in (59)

$$\left(1 + \frac{\xi\eta\sigma}{1-\xi}\right)\frac{\lambda_\kappa}{\psi}\kappa_t - \frac{\eta}{1-\xi}\left[\lambda_c(c_t^b - c_t^s) + (\zeta_h + \xi\alpha_h)\lambda_h(h_t^b - h_t^s)\right] = 0, \tag{67}$$

where

$$\zeta_h \equiv \frac{\omega + \beta_s + \xi(\tilde{\mu} - \sigma_h \omega)}{\omega + \beta + \xi\tilde{\mu}}.$$

Differently from the case of flexible prices, capital requirements respond not only to the housing gap but also to the consumption gap. The intuition is that a combination of capital requirements, the consumption gap, and the housing gap is proportional to output and inflation (compare 66 and 67). Therefore, by responding also to the consumption gap, the policymaker internalises the consequences of monetary policy for macro-prudential policy.

Notice that we can alternatively write the last targeting rule as

$$\frac{\eta}{1-\xi}\zeta_h\lambda_h(h_t^b - h_t^s) - \frac{\lambda_\kappa}{\psi}\kappa_t \equiv \mathcal{H}_{\kappa t} = \frac{\eta}{1-\xi}\left[\xi\sigma\frac{\lambda_\kappa}{\psi}\kappa_t - \lambda_c(c_t^b - c_t^s) - \xi\alpha_h\lambda_h(h_t^b - h_t^s)\right]$$

This expression is useful because we can then write the last targeting rule from (63) as

$$\mathcal{H}_{\kappa t} = \frac{\tilde{\mu}\zeta_h}{\sigma_h \omega}\lambda_h(h_t^b - h_t^s) + \beta\mathbb{E}_t\mathcal{H}_{\kappa t+1}. \tag{68}$$

According to this rule, macro-prudential policy needs to take into account current and future expected housing gaps in order to find the appropriate response to current developments.

F Quantitative Experiment Details

In this Appendix, we provide more details of the approach used to compute the simulations in Section 4.

F.1 Incorporating News Shocks

The quantitative experiment we consider in the main text assumes that agents' expectations about future housing demand shocks evolves over time. To solve the model we therefore need to incorporate housing demand news shocks into the model.

We do this by defining the forcing process for u_t^h as:

$$u_t^h = \rho_h u_{t-1}^h + \epsilon_t^h + \nu_{t-1}^{h,1} \tag{69}$$

where $\nu_t^{h,j}$ captures the effect of news shocks j periods ahead in period t. Thus $\nu_{t-1}^{h,1}$ affects the housing demand process in period t because it captures the total effects of shocks that were anticipated in the previous period $(t-1)$ to arrive in the current period (t). So $\nu_t^{h,j}$ captures the total effect of previously revealed news shocks anticipated to arrive in period t and can therefore be defined recursively as:

$$\nu_t^{h,j} = \epsilon_t^{h,j} + \nu_{t-1}^{h,j+1} \tag{70}$$

for $j = 1, \ldots, J-1$ and with $\nu_t^{h,J} = \epsilon_t^{h,J}$.

There are other ways to incorporate news shocks into models. Indeed the Anderson and Moore (1985) algorithm that we use to solve for the rational expectations equilibrium has a particularly convenient representation (see below). However, incorporating the news shocks into the state vector of the model ensures that they are correctly accounted for when computing the optimal discretionary policy.

F.2 Optimal Discretion

To solve for optimal discretionary policies we use the algorithm developed by Dennis (2007). This algorithm accepts a model written in terms of non-policy variables $\widehat{x}$ and policy instruments r of the form:

$$H_{F,x}\mathbb{E}_t\widehat{x}_{t+1} + H_{F,r}\mathbb{E}_t r_{t+1} + H_{C,x}\widehat{x}_t + H_{C,r}r_t + H_B\widehat{x}_{t-1} = \widehat{\Psi}\epsilon_t \tag{71}$$

where ϵ is the vector of innovations to the model. These innovations are assumed to be mean zero and identically and independently distributed.

The policymaker is assumed to set policy to minimise a loss function given by:

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta\left[\widehat{x}_t'W\widetilde{x}_t + r_t'Qr_t\right]$$

and Dennis shows that the optimal policy satisfies the following first order condition:

$$Qr_t + H_{C,r}'\left(D'\right)^{-1}P\widetilde{x}_t = 0 \tag{72}$$

where

$$D \equiv -H_{C,x} - H_{F,x}J_1 - H_{F,r}G_1$$
$$P = W + \beta G_1'QG_1 + +\beta J_1'PJ_1$$

and the G_1 and J_1 matrices are from the Markovian solutions for the policy and non-policy variables respectively. That is, in equilibrium, the decision rules are given by $\widehat{x}_t = J_1\widehat{x}_{t-1} + J_2\epsilon_t$ and $r_t = G_1r_{t-1} + G_2\epsilon_t$. Dennis (2007) provides a simple iterative algorithm for solving for the matrices G_1, G_2, J_1, J_2. With those solutions in hand, P can be found using a Lyapunov equation solver.

F.3 Dealing with Occasionally Binding Constraints

The approach was set out by Holden and Paetz (2012) and is convenient in our case.

F.3.1 The Model and the Rational Expectations Solution

The starting point is a model of the policy and non-policy variables, which we stack together as $x_t \equiv [\widehat{x}_t' \;\; r_t']'$. This model is written as:

$$H_F \mathbb{E}_t x_{t+1} + H_C x_t + H_B x_{t-1} = \Psi \epsilon_t + \Psi_\delta \delta_t \tag{73}$$

and is formed by stacking the equations (71) and the optimal policy rule (72), so that:

$$H_F = \begin{bmatrix} H_{F,x} & H_{F,r} \\ 0 & 0 \end{bmatrix} \quad H_C = \begin{bmatrix} H_{C,x} & H_{C,r} \\ H_{C,r}' \left(D'\right)^{-1} P & Q \end{bmatrix} \quad H_B = \begin{bmatrix} H_{B,x} & 0 \\ 0 & 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} \widehat{\Psi} \\ 0 \end{bmatrix}$$

The new component to the model is a vector of 'shocks' δ that are introduced in order to impose the occasionally binding constraints. These shocks are added to the model equations which do not hold when the occasionally binding constraints are binding. For example, consider the case in which the monetary policymaker pursues a 'flexible inflation targeting' mandate, with no macro-prudential policy in place. That is, the policymaker minimises $\mathcal{L}_0^{FIT}$ defined in the main text. As noted there, the optimal discretion first order condition (72) can be written as (22). The nominal interest rate required to deliver (22) may be negative, which would violate the zero lower bound. To impose the zero lower bound, we augment (22) with a 'shock' $\delta_{i,t}$ to give:

$$x_t + \gamma \lambda_\pi \pi_t + \delta_{i,t} = 0 \tag{74}$$

When the zero bound does not bind, $\delta_{i,t} = 0$ and (74) collapses to (22). When the zero bound binds, $\delta_{i,t}$ is chosen so that the nominal interest rate is equal to the lower bound.

A similar approach is used to account for the fact that the borrowing limit may not bind. The model is augmented with the following conditions:

$$\begin{aligned} d_t^{gap} &= d_t^b - \gamma_d \left(d_{t-1}^b - \pi_t\right) - (1-\gamma_d)\left(\theta_t + q_t + h_t^b\right) \\ d_t^{gap} + \delta_{d,t} &= 0 \end{aligned}$$

where the first equation defines the gap between the level of debt (d_t^b) and the the borrowing constraint at equality $\gamma_d \left(d_{t-1}^b - \pi_t\right) + (1-\gamma_d)\left(\theta_t + q_t + h_t^b\right)$. The second equation introduces the 'shock' $\delta_{d,t}$. When the borrowing constraint binds, $\delta_{d,t} = 0$ so that $d_t^{gap} = 0$ and the level of debt is determined by the borrowing constraint. When the borrowing constraint is slack, $\delta_{d,t} > 0$ is chosen so that the multiplier on the constraint is zero ($\tilde{\mu}_t = -\tilde{\mu}^{ss}$). In that case, $d_t^{gap} < 0$ and the level of debt is less than the borrowing constraint.

Our approach shares some similarities with the 'OccBin' approach developed by Guerrieri and Iacoviello (2015). For example, when the ZLB binds, the targeting criterion (22) does not form part of the model. Instead, the shock $\delta_{i,t}$ is chosen to enforce that the interest rate satisfies the zero bound: in effect, we replace (22) with the equation $i_t = i^{ZLB} < 0$ where i^{ZLB} is the lower bound. This is analogous to the OccBin approach of defining different sets of model equations that apply when constraints are or are not binding. One advantage of our approach is that it scales easily as the number of occasionally binding constraints grows.[24] As we show below, our approach also allows us to check for the uniqueness of the solution.

Our approach requires us to solve for the values of δ_t (with $t = 1, \dots$) that impose the occasionally binding constraints. To do that, we will use the rational expectations solution of the model (73) which the Anderson and Moore (1985) algorithm delivers as:

$$x_t = Bx_{t-1} + \Phi\epsilon_t + \sum_{i=0}^{\infty} F^i \Phi_\delta \mathbb{E}_t \delta_{t+i} \tag{75}$$

where B, F, Φ and Φ_δ are functions of the coefficient matrices H_F, H_C, H_B, Ψ and Ψ_δ in (73). The solution in (75) is valid for any expected shock sequence $\{\delta_{t+i}\}_{i=0}^{\infty}$.[25] This is important because our solution must cope with the fact that the equilibrium in period t may be affected by the expectation that occasionally binding constraints are binding in period(s) $s > t$.

F.3.2 The Baseline Simulation

To simulate the model we first assume that none of the occasionally binding constraints binds. This is our 'baseline simulation'. To produce it we set $\delta_t = 0, \forall t$ and then from a given initial condition x_0 and a realization of the shocks ϵ_1 we compute $x_t = Bx_{t-1} + \Phi\epsilon_t$ for $t = 1, \dots, H$ for some simulation horizon H.[26]

With the baseline simulation in hand, we then check whether it violates the assumption that the constraints never bind. So, for example, we check whether the implied trajectory of the multiplier on the borrowing constraint is always positive ($\tilde{\mu}_t > -\tilde{\mu}^{ss}, \forall t$) and whether the path of the policy rate is always positive ($i_t > i^{ZLB}, \forall t$). If we find that any of these assumptions is violated in the baseline, then we need to invoke a quadratic programming procedure to ensure that the occasionally binding constraints are enforced.

F.3.3 Imposing the Occasionally Binding Constraints (OBC)

The OBC can be represented as inequality constraints on a set of 'target variables' which we will denote as τ. In our case, τ would include the policy rate and the multiplier on the borrowing constraint. To impose the OBC, we will solve for a set of shocks $\{\delta_t\}_{t=1}^{H}$ that impose the OBC. The simulation horizon H can be chosen to be arbitrarily large (so that the solution will be close to steady state by the end of the simulation).

[24]Incorporating N occasionally binding constraints using OccBin requires specifying 2^N alternative sets of model equations, whereas in our approach we need to add N 'shocks' (and possibly up to N auxiliary equations/variables such as d^{gap}).

[25]As long the shocks do not increase at a rate faster than (the inverse of) the maximum eigenvalue of F.

[26]Note that ϵ_1 can incorporate news shocks (that arrive in period $s > 1$) if they are encoded into the state vector as described in Section F.1.

The approach is based on the insight that the effect of the δ shocks can be simply added to the baseline simulation, given the linearity of the model. Inspection of (75) reveals that the effect of the fundamental and dummy shocks enter linearly. So to find the set of dummy shocks that ensure that the target variables satisfy the OBC, we solve for a set of shocks that, when added to the baseline simulation will achieve this. To do so, we need to be able to record the impact of δ shocks at all horizons $t = 1, \ldots, H$ on the target variables in all periods $t = 1, \ldots, H$.

Let S_τ be a selector matrix that selects the target variables from the vector of endogenous variables. Thus:

$$\tau_t = S_\tau x_t \tag{76}$$

Consider now the effects of the δ shocks $\{\delta_t\}_{t=1}^{H}$ on the endogenous variables in period 1 of the simulation. This is given by:

$$\widehat{x}_1 = \sum_{i=0}^{H-1} F^i \Phi_\delta \delta_{1+i}, \tag{77}$$

which captures the fact that in period 1 all of the shocks occur in (present and) future periods. The effects on the target variables are given by $\widehat{\tau}_1 = S_\tau \widehat{x}_1$.

In period 2, we can use the RE solution to note that the effects on endogenous variables are:

$$\widehat{x}_2 = B\widehat{x}_1 + \sum_{i=0}^{H-2} F^i \Phi_\delta \delta_{2+i}, \tag{78}$$

and from the expression for $\widehat{x}_1$, we can write:

$$\widehat{x}_2 = B\sum_{i=0}^{H-1} F^i \Phi_\delta \delta_{1+i} + \sum_{i=0}^{H-2} F^i \Phi_\delta \delta_{2+i}. \tag{79}$$

This step provides a recursive scheme for building a matrix that maps the effects of shocks to the dummy shocks in periods $t = 1, \ldots, H$ to the target variables in each period. The first (block) row of this matrix can be found by expanding (77):

$$\widehat{\tau}_1 = \left[\begin{array}{ccccc} S_\tau \Phi_\delta & \ldots & S_\tau F^{k-1}\Phi_\delta & \ldots & S_\tau F^{H-1}\Phi_\delta \end{array} \right] \left[\begin{array}{c} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{array} \right]. \tag{80}$$

The second row is built by using equation (79) to multiply the coefficients in the first row by

B and then adding the coefficients on shocks that arrive from period 2 onwards:

$$\widehat{\tau}_2 = \begin{bmatrix} S_\tau B \Phi_\delta & \dots & S_\tau B F^{k-1}\Phi_\delta + S_\tau F^{k-2}\Phi_\delta & \dots & S_\tau B F^{H-1}\Phi_\delta + S_\tau F^{H-2}\Phi_\delta \end{bmatrix} \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{bmatrix},$$

and this can be applied for each row in turn.

This scheme implies that we can write the mapping from the dummy shocks to the target variables as:

$$\mathcal{T} = \mathcal{M}\mathcal{D}, \tag{81}$$

where

$$\mathcal{T} = \begin{bmatrix} \widehat{\tau}_1 \\ \vdots \\ \widehat{\tau}_k \\ \vdots \\ \widehat{\tau}_H \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{bmatrix}, \tag{82}$$

and the rows of $\mathcal{M}$ are built using the recursive scheme described above.

Notice that if the number of OBCs is n, then τ and δ are $n \times 1$ vectors so that $\mathcal{T}$ and $\mathcal{D}$ are $(nH) \times 1$. The mapping we have derived records the effects of all δ shocks on all target variables. This reflects the fact that there may be interactions between the different OBCs. For example, if the borrowing constraint happens to be slack in a period of weak growth, economic outcomes may be better than otherwise and so the ZLB becomes non-binding. To correctly capture these types of effects we need to incorporate the effects of shocks that implement each OBC on all target variables.

To incorporate the bounds on the OBCs, we compute the vector $\widehat{\mathcal{T}}$ as the deviation of the target variables from their constraint values. This is just a normalization, but it is useful in setting up the quadratic programming problem (because it allows us to incorporate a contemporary slackness condition easily). To do this, we simply record the relevant rows of the baseline simulation $\{x_t\}_{t=1}^{H}$ and subtract the value of the constraints. This normalization implies that if the baseline simulation implied $\widehat{\mathcal{T}} > 0$, then the baseline solution, which assumes that the OBCs never bind, would be correct.

We can now set up a quadratic progamming problem to solve for $\mathcal{D}$:

$$\min \quad \frac{1}{2}\mathcal{D}'\left(\mathcal{M}+\mathcal{M}'\right)\mathcal{D}+\widehat{\mathcal{T}}'\mathcal{D} \tag{83}$$

$$\text{subject to: } \widehat{\mathcal{T}}+\mathcal{M}\mathcal{D}\geq 0 \tag{84}$$

$$\mathcal{D}\geq 0 \tag{85}$$

The problem in equations (83)–(85) can be understood as follows. The constraint (84) ensures that the OBCs are respected. $\widehat{\mathcal{T}}$ is the baseline simulation for the target variables, measured relative to the constraint values. $\mathcal{M}\mathcal{D}=\mathcal{T}$ is the marginal effect of the δ shocks $\mathcal{D}$ on the target variables. So $\widehat{\mathcal{T}}+\mathcal{M}\mathcal{D}$ is the path of the target variables measured relative to their constraints after the δ shocks have been applied: requiring this to be positive implies that the constraints are respected.

The constraint (85) requires that the δ shock values used to impose the constraints are positive. This requirement ensures that the OBCs are truly binding. To see why this is important, suppose for a moment that monetary policy is determined by a Taylor rule including a δ shock to enforce the ZLB. Consider a simulation in which there is an initial negative shock to demand that causes the Taylor rule to prescribe a negative value for the policy rate in the first few periods of the baseline simulation. Now suppose that we seek δ shocks to the Taylor rule to ensure that the ZLB is respected. One solution could be to apply *negative* future shocks to the policy rule that push future rates lower than the baseline simulation: this could be enough to boost demand in the near term such that the ZLB never binds (and so constraint (84) is respected as a strict inequality).

Finally, note that the minimand (83) can be expanded as follows:

$$\begin{aligned}\frac{1}{2}\mathcal{D}'\left(\mathcal{M}+\mathcal{M}'\right)\mathcal{D}+\widehat{\mathcal{T}}'\mathcal{D} &= \frac{1}{2}\mathcal{D}'\mathcal{M}\mathcal{D}+\frac{1}{2}\mathcal{D}'\mathcal{M}'\mathcal{D}+\frac{1}{2}\widehat{\mathcal{T}}'\mathcal{D}+\frac{1}{2}\mathcal{D}'\widehat{\mathcal{T}} \\ &= \frac{1}{2}\mathcal{D}'\left(\mathcal{M}\mathcal{D}+\widehat{\mathcal{T}}\right)+\frac{1}{2}\left(\mathcal{M}\mathcal{D}+\widehat{\mathcal{T}}\right)'\mathcal{D},\end{aligned}$$

where the first line exploits the fact that $\widehat{\mathcal{T}}'\mathcal{D}$ is a scalar and the second line collects terms. The minimand is therefore analogous to a contemporary slackness condition: it achieves a minimum of zero when $\mathcal{D}=0$ or $\widehat{\mathcal{T}}+\mathcal{M}\mathcal{D}=0$.

The above discussion assumed that the final constraint $\mathcal{D}>0$ is economically sensible given the model at hand. For this to be true we require that an anticipated positive δ shock that arrives j periods ahead will be expected to increase the bounded variable in period j. This seems like it should be automatically satisfied, but the interaction of lead/lag relationships in medium-scale DSGE models means that it need not be satisfied. One example is the 'reversed sign' responses of some DSGE models to monetary policy shocks in the distant future.[27]

It is straightforward to check whether the model suffers from this problem by inspecting the

[27]In this case a positive shock to the monetary policy rule in the distant future causes a contraction today because of forward looking behavior. The monetary policy reaction function prescribes a near-term loosening in response to the contractionary effect of the future policy tightening. If the variables that enter the policy rule and/or the policy rate itself are sufficiently inertial, we may observe an equilibrium in which the policy rate is lower in period j because the negative effects on the arguments of the rule outweigh the positive effects of the shock.

signs of the diagonal elements of the $\mathcal{M}$ matrix. If they are all positive, then we can apply the algorithm as presented above. If some are negative, we need to amend the $\mathcal{D} \geq 0$ constraint to flip the sign applied to the relevant elements of $\mathcal{D}$.

Another issue is that the quadratic programming problem has a unique solution only if the matrix $(\mathcal{M} + \mathcal{M}')$ is positive semi-definite. A sufficient condition for the matrix to be positive semi-definite is for its eigenvalues to be non-negative which can be easily checked.

Finally, there is no guarantee that a solution exists. For very large shocks, the overarching assumption that the model returns to 'normal' in a finite period of time may be violated (for example, the model may get stuck in a deflation trap). Non-existence is likely to be a problem when there is a strong feedback between the OBCs. Again, in practice this can be checked by ensuring that the δ shocks are zero at the end of the simulation horizon H.